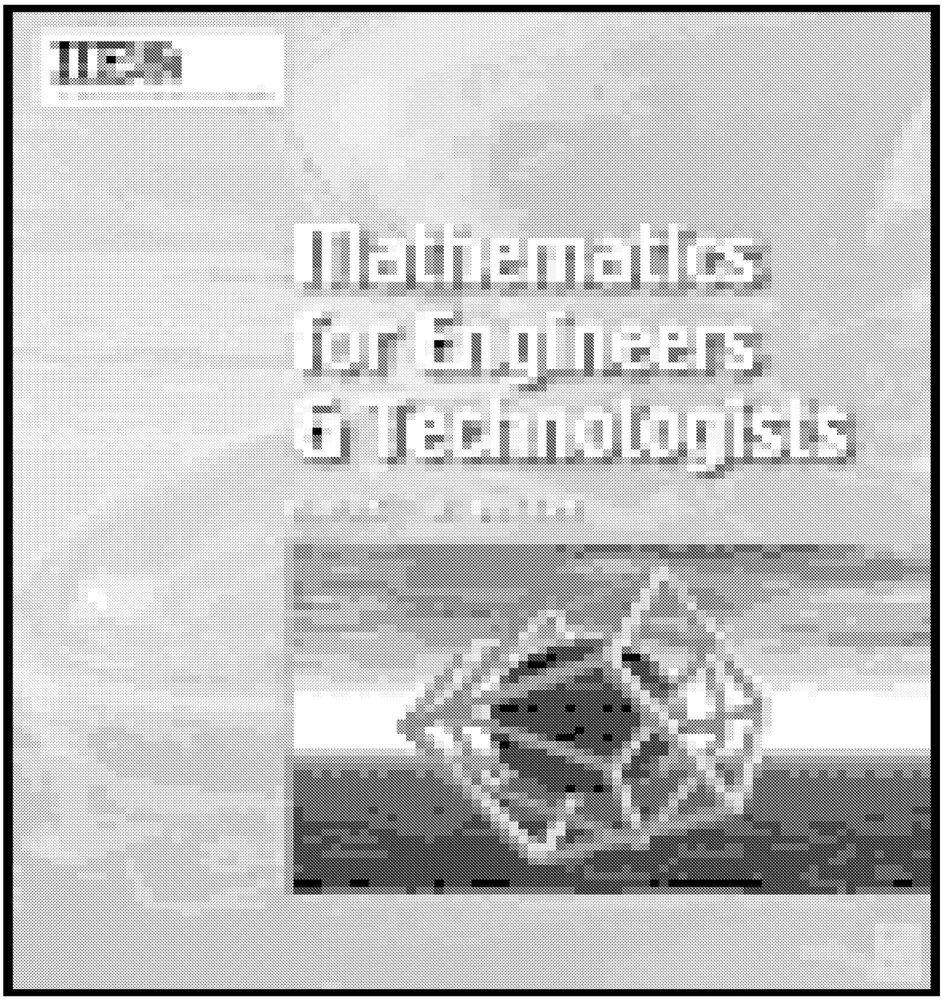
Mathematics for Engineers and Technologists

by Huw Fox, W. Bolton



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Series Preface

'There is a time for all things: for shouting, for gentle speaking, for silence; for the washing of pots and the writing of books. Let now the pots go black, and set to work. It is hard to make a beginning, but it must be done' – Oliver Heaviside, *Electromagnetic Theory*, Vol 3 (1912), Ch. 9 'Waves from moving sources - Adagio, andante, Allegro Moderato.'

Oliver Heavyside was one of the greatest engineers of all time, ranking alongside Faraday and Maxwell in his field. As can be seen from the above excerpt from a seminal work, he appreciated the need to communicate to a wider audience. He also offered the advice 'So be rigorous: that will cover a multitude of sins. An do not frown.' The series of books that this prefaces takes up Heavyside's challenge but in a world which is quite different to that being experienced just a century ago.

With the vast range of books already available covering many of the topics developed in this series, what is this series offering which is unique? I hope the next few paragraphs help to answer that; certainly no one involved in this project would give up their time to bring these books to fruition if they had not thought that the series is both unique and valuable.

The motivation for this series of books was born out of the desire of the UK's Engineering Council to increase the number of incorporated engineers graduating from Higher Education establishments, and the Institution of Incorporated Engineers' (IIE) aim to provide enhanced services to those delivering Incorporated Engineering Courses. However, what has emerged from the project should prove of great value to a very wide range of courses within the UK and internationally – from Foundation Degrees or Higher Nationals through to first year modules for traditional 'Chartered' degree courses. The reason why these books will appeal to such a wide audience is that they present the core subject areas for engineering studies in a lively, student-centred way, with key theory delivered in real world contexts, and a pedagogical structure that supports independent learning and classroom use.

Despite the apparent waxing of 'new' technologies and the waning of 'old' technologies, engineering is still fundamental to wealth creation. Sitting alongside these are the new business focused, information and communication dominated, technology organisations. Both facets have an equal importance in the health of a nation and the prospects of individuals. In preparing this series of books, we have tried to strike a balance between traditional engineering and developing technology.

The philosophy is to provide a series of complementary texts which can be tailored to the actual courses being run – allowing the flexibility for course designers to take into account 'local' issues, such as areas of particular staff expertise and interest, while being able to demonstrate the depth and breadth of course material referenced to a common framework. The series is designed to cover material in the core texts which approximately corresponds to the first year of study with module texts focusing on individual topics to second and final year level. While the general structure of each of the texts is common, the styles are quite different, reflecting best practice in their areas.

Another set of factors which we have taken into account in designing this series is the reduction in contact hours between staff and students, the evolving responsibilities of both parties and the way in which advances in technology are changing the way study can be, and is, undertaken. As a result, the lecturers' support material which accompanies these texts, is paramount to delivering maximum benefit to the student.

It is with these thoughts of Voltaire that I leave the reader to embark on the rigours of study:

'Work banishes those three great evils: boredom, vice and poverty'

Alistair Duffy
Series Editor
De Montfort University, Leicester, UK

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Introduction: why mathematics?

Mathematics is an essential tool for the engineer and technologist. As an illustration, consider a number of simple situations:

- ***A beam***

  A uniform horizontal beam rests on supports at each end and loads placed at its mid point. How does the deflection of the beam from the horizontal at its mid point depend on the applied load? Can we develop a mathematical relationship which will enable us to predict the deflection for a given load? We might need such a relationship in order to be able to consider the design of a simple plank bridge across a stream, or the elements in a more complex truss bridge.

  See **Chapter 1** for a discussion of mathematical relationships; when one quantity is dependent on another it is said to be a function of it.

  We might develop such a relationship by conducting an experiment in which we measure the deflections for a number of loads and so develop an empirical relationship which fits the results. This could involve plotting a graph between the force and deflection and from the 'shape' of the graph determining a relationship. This requires an understanding of graphs and, in particular, straight line graphs. If the graph between two quantities is not a straight line then engineers use 'tricks' to persuade the graph to become straight line because straight line graphs enable relationships to be most easily discerned.

  See **Chapter 3** for the determination of relationships from graphs by 'persuading' them to become straight line graphs.

  We might, however, develop the relationship from a consideration of how beams behave when subject to loads and so end up with a more general relationship which we can apply to other beams. In developing such a relationship we would use algebra, i.e. the quantities such as force and deflection are represented by letters such as F and y, and so we need to be able to manipulate algebraic expressions. In fact, the basic expression for elasticity involves a differential equation, i.e. an equation involving terms concerned with rates of change, and so to derive the relationship for the deflection we need to be able to solve such an equation.

  See **Chapters 4 and 5** for a discussion of calculus and the solution of differential equations.

- ***An oscillation***

A loaded vertical spring is set in oscillation; what are the factors determining the frequency of the oscillation? We might carry out an experiment involving different loads on a spring, and also try a number of different springs. The relationship between frequency and load can be determined by plotting a graph. However, if we just plot frequency against load we obtain a non-linear graph and it is not easy to see the relationship. If we plot the logarithm of the frequency against the logarithm of the load then a straight line graph is obtained and the relationship can easily be discerned.

See **Chapter 1** for a discussion of logarithms and log graphs.

Alternatively, a more general relationship might be found from the theory of oscillations. We might develop the theory by considering a model of oscillations in which the variation of displacement with time is represented by how the vertical height of a rotating radius varies with time when the radius rotates with a constant angular velocity.

See **Chapter 1** for an introduction to phasors and **Chapter 2** for a more detailed consideration.

The relationships devised for this simple system of an oscillating spring can, however, provide a basic introduction to the consideration of much more complex oscillations.

See **Chapter 6** for a discussion of how oscillations can be represented by differential equations and **Chapter 7** for how we can use the Laplace transforms to simplify the handling of such equations.

- ***An a.c. electrical circuit***

A simple electrical circuit is set up with an a.c. voltage being applied to a series circuit of a capacitor and a resistor. To develop a theory which enables the circuit current to be determined by different values of capacitance and resistance and also for different frequency alternating currents, an approach based on a consideration of phasors is generally used. A phasor is used to model in its length the amplitude or root-mean-square value of a voltage or current and by its initial angle the phase. By using such models, the analysis of a.c. circuits is simplified and we do not need differential equations. The phasors can be drawn and added or subtracted graphically. However, an algebraic method is to use complex numbers.

See **Chapter 1** for an introduction to phasors and **Chapter 2** for a more detailed consideration.

We might have the complication with such a circuit that the alternating voltage is not sinusoidal. In such a case we can deal with the circuit analysis by representing the signal as a series of sinusoidal terms.

See **Chapter 7** for a discussion of the Fourier series.

- ***Programmable logic controller (PLC)***

PLCs are widely used for control systems. Such controllers can be easily programmed to carry out such operations as switching on motors when sensor A and sensor B both give ON signals or perhaps when either sensor A or sensor B gives an ON signal. Thus a central heating system controller needs to switch the pump motor on when either the temperature sensor on the hot water tank gives an ON signal or the room temperature sensor gives an ON signal. Such control systems require a consideration of basic logic systems.

See **Chapter 8** for a discussion of logic gates.

- ***Measurements***

  Engineering involves making measurements. With any measurement there is inevitably some associated error. To estimate and handle such errors in calculations based on the measurements, we need an understanding of basic statistics.

See **Chapter 9** for a discussion of statistics and the handling of errors.

In considering the elasticity of a beam we form mathematical models of the real world situation. Thus, in the case of the beam, we make a number of simplifying assumptions, such as the deflections are small, the beam is thin and of uniform cross-section. With such simplifications we can produce a model. Real beams may behave differently because the assumptions are not valid but our mathematical model provides working relationship. In some situations we develop what are obviously models of systems. For example, we might represent the behaviour of a car suspension and wheel as a mass with a spring and a damper. In considering a motor we might be able to represent it by the model of an inductor in series with a resistor and a source of e.m.f. for the back e.m.f. of the motor.

See **Chapter 3** for an introductory discussion of mathematical models.

As the above examples illustrate, we need mathematics to be able to solve engineering problems. Mathematics, however, is not a tool that you can pick up and use without an understanding of the principles behind its development and its limitations. Thus, in this book, the principles behind the mathematics are explored and the book is more than just a collection of 'cookbook techniques' which can be used for particular situations. Such a 'cookbook' approach presents problems if you encounter, as engineers inevitably do, a new situation. The aim of this book is:

> *To enable the reader to understand the principles of the mathematics and acquire the ability to use it in engineering.*

With that aim in mind, we hope you will enjoy the book and wish you well in your studies.

1 Functions

Summary

Engineers, whether electrical, electronic, mechatronic or mechanical, are concerned with expressing relationships between physical quantities clearly and unambiguously. This might be the relationship between the displacement of an oscillating object and time, or perhaps the amplitude of an a.c. voltage and time. This chapter is about how we can represent such relationships in mathematical terms, taking the opportunity to revise some basic mathematics in the process. This does not mean that it is not important to clearly explain in words what relationships there are between quantities but rather to supplement the written word by using a system that is both clear, unambiguous, and internationally understood, thereby removing the possibility of misinterpretation.

Objectives

By the end of this chapter, the reader should be able to:

- understand the concept of a function for relating quantities within engineering disciplines;
- use functions, and their notation, to describe relationships;
- manipulate and evaluate algebraically simple function expressions, including inverses;
- use graphs to express functions;
- express cyclic functions in terms of sine and cosine functions;
- know, and use, the relationships between sine, cosine and tangent ratios;
- describe waveforms in the general format $R \sin(\omega t \pm a)$;
- use exponential and logarithmic functions;
- use hyperbolic functions.

1.1 Introduction to functions

As you embark on studying engineering, whether electrical, electronic, mechatronic or mechanical, it will become apparent that equations are used to describe relationships between physical quantities and are more clear and unambiguous than the written word on its own. This does not mean that it is not important to clearly explain relationships in words but rather there is a need to supplement the written word by using equations. In any discussion

of relationships between physical quantities, the term function is likely to be encountered. So what is a function?

So what is a function?

Let's commence our explanation of the term *function* by discussing a simple example.

We are all familiar with springs, whether they are those we find inside some pens, the springs holding the tremolo block in position on a classic Fender stratocaster guitar or the suspension springs on a car. Consider, therefore, a simple vertical spring dangling from a clamp with a mass pan attached to its lower end (Figure 1.1). Using such a system we can perform a simple experiment, adding masses to the pan and determining the relationship between the force exerted by the masses and the resulting extension of the spring. The extension is measured from the datum line of the system when in stable equilibrium before we start adding masses and recording extensions. The results of such an experiment might be of the form shown in Table 1.1.

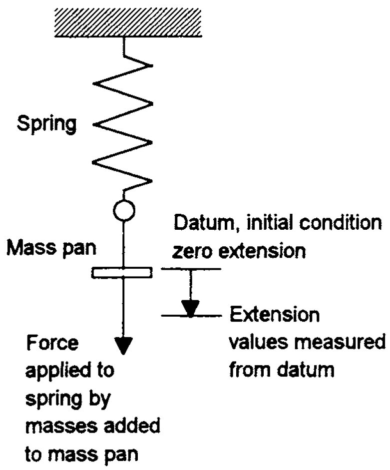


Figure 1.1 *Simple spring with mass pan, the figure indicating the two physical quantities, namely the applied force and the resulting extension, that we are interested in determining the relationship between. Note, adding masses of 120 g results in force increments of about 1.2 N.*

Table 1.1 *Force–extension values for simple spring example*

Force F applied to spring, N	Corresponding extension e of spring, m
0.0	0.00
1.2	0.01
2.4	0.02
3.6	0.03
4.8	0.04
6.0	0.05

From Table 1.1, we can assume that there is some relationship or connection between the values of the force F acting on the spring and the corresponding observed extension e values. We can say:

the extension e depends on the corresponding applied force F [1]

Statement [1] is an inferred relationship between an observed measurement e and a varied quantity, in this case the applied force F. We can therefore call F the *independent* quantity, sometimes referred to as the *argument*, from which a *dependent* result is obtained. We can restate statement [1] as:

the extension e is a function of the applied force [2]

A *function* is a relationship which has for each value of the independent variable a unique value of the dependent variable. Statement [2] can be written as:

$e = f(F)$

> **Key point**
>
> Quantities which vary, like the force and extension in the case of the spring system, are called *variables* and the *dependent variable*, the extension in this case, is the one whose values depend on the values of the *independent variable* or *argument*, in this case the force.

> **Key point**
>
> A function is a relationship which has for each value of the independent variable a unique value of the dependent variable.

This means exactly the same thing as statement [2] but is just easier and more concise to write. The 'f' simply is shorthand for 'function of'. Note that $f(F)$ does not mean a variable f multiplied by F. When we are dealing with a number of different functions it is customary to use different letters for the function label, e.g. we might use $y = f(x)$ and $z = g(x)$.

Tables, graphs and equations to define functions

Key points

With $y = f(x)$ we have each value of y associated with an x value. We can plot such data points on a graph having x and y axes, the y-axis running vertically and the x-axis running horizontally and at 90° to the y-axis (Figure 1.2). The point of intersection of the two axes is called the *origin*; at this point y has the value 0 and x = 0. Values of x which are positive are plotted to the right of the origin, negative values to the left. Positive values of y are plotted upwards from the origin, negative values downwards.

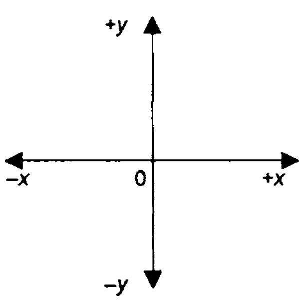


Figure 1.2 *Graph axes*

The statement $e = f(F)$ merely tells us that the extension e is a function of the applied force F and does not describe the actual relationship between them. The data in Table 1.1 is one way we describe the relationship. If we know the force is 3.6 N then the extension must be 0.03 m.

However, a pattern can be seen from an inspection of the results in Table 1.1: if we double the applied force then the resulting extension doubles, if we treble the applied force the extension trebles and if we quadruple the applied force the extension quadruples. We can say, at least over the range of values we have observed:

the extension e is directly proportional to the applied force F

and we can write this as:

$$e \propto F \qquad [3]$$

The symbol $\propto$ simple means (or is shorthand for) 'proportional to'.

We can also see how the extension depends on the applied force by plotting a graph. If we plot a graph of the force values in Table 1.1 against the corresponding extension values we obtain the straight line graph shown in Figure 1.3. The straight line passes through the origin. This is a characteristic of relationships when one quantity is directly proportional to the other. The graph is thus one way of defining the functional relationship.

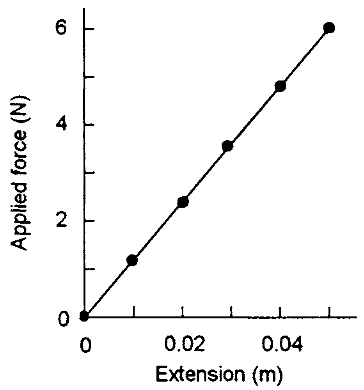


Figure 1.3 *Force–extension graph for the spring experiment values given in Table 1.1*

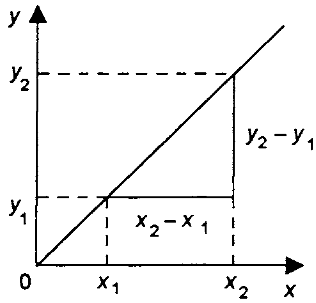


Figure 1.4 *Generalised straight line graph witth the slope being* $(y_2 - y_1)/(x_2 - x_1)$

We can use the graph in Figure 1.3 to determine an equation relating the extension to the applied force and so give another way of defining the functional relationship. The graph is a straight line and so has a constant slope (or gradient), the slope being defined in the same way as we define the slope of a road, i.e. as the change in vertical height of the line over a given horizontal distance (Figure 1.4). We can compute the slope by choosing a pair of values of force and extension, e.g. force 60 N, extension 0.05 m; force 1.2 N, extension 0.01 m, and so obtain:

$$\text{slope} = \frac{6.0 - 1.2}{0.05 - 0.01} = \frac{4.8}{0.04} = 120$$

Since the straight line passes through the origin, this tells us that the force in newtons is, over the interval that has been considered, 120 times the extension in metres. We can write this as:

$$F = 120e \qquad [4]$$

The constant term, i.e. the 120 N/m, is called the *constant of proportionality* linking F to the corresponding e values.

To summarise: we can define the functional relationship between two variables by:

- a set of results (as in Table 1.1),
- a graph (as in Figure 1.3),
- an equation (as in equation [4]).

Note that to define a function $y = f(x)$ completely, we must define the range of values of x over which the definition is true. This is called the *domain* of the function.

Example

If y is a function of x and the relationship is defined by the equation $y = 4x^2$, what is the value of y when $x = 2$?

We have:

$y = f(x) = 4x^2$

and so:

$y = 4(2^2) = 4 \times 2^2 = 16$

Therefore, $y = 16$ when $x = 2$. Substituting $x = 2$ into the original functional equation, we can write:

$y = f(x) = f(2) = 16.$

Example

If y is a function of x and the relationship is defined by the equation $y = 12x^2 + 3x + 6$, what is the value of y when $x = 1$?

We have:

$$y = f(x) = 12x^2 + 3x + 6$$

and so:

$$y = 12(1^2) + 3(1) + 6 = 12 + 3 + 6 = 21$$

Therefore, $y = 21$ when $x = 1$. Substituting $x = 1$ into the original functional equation, we can write:

$$y = f(x) = f(1) = 21$$

Equations and functions

Functions may, as mentioned earlier, be defined using equations. Equations give the instructions for calculating the dependent variable of functions for values of the independent variable. For example, for an ohmic resistor the potential difference V across it is a function of the current I through it, i.e.

$$V = f(I)$$

The equation defining the functional relationship (Ohm's law) is $V = RI$, where R is the constant if proportionality connecting the variable V with the variable I, thereby defining their unique relationship. So given a value for the current we can use the equation to obtain a value of the potential difference. Thus, when $R = 10\ \Omega$ and $I = 2$ A we have:

$$V = f(2) = 20 \text{ V}.$$

For an object freely falling from rest, the distance fallen s is a function of the time t for which it has been falling, i.e. $s = f(t)$. The defining equation is $s = ½at^2$, where a, the acceleration, is a constant. The acceleration is the acceleration due to gravity g and so we can write the defining equation as $s = ½gt^2$. Thus, given a value for the time we can use the equation to obtain a value for the distance fallen. If we assume that g has a value of 9.8 m/s^2, then for a time of 3 s;

$$s = f(3) = \tfrac{1}{2} \times 9.8 \times 3^2 = 44.1 \text{ m}$$

Note that a function may be defined by several equations, with each giving the instructions for calculating the dependent variable for different values of the independent variable. For example, for

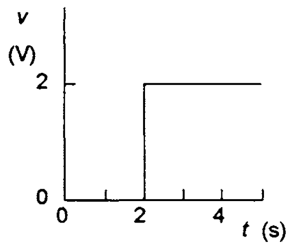


Figure 1.5 *Step voltage*

the voltage signal shown in Figure 1.5, a so-called *step voltage*, we have $v = f(t)$ and the relationship

$v = 0$ for t between 0 and 2 s, i.e. $0 \le t < 2$
$v = 2$ V for t greater than 2 s, i.e. $2 \le t$

The value of $v = f(1)$ is thus 0, while the value of $v = f(4)$ is 2 V.

Functions may of course get quite complex! For example, the natural frequency of transverse vibration of a cantilever is described by the equation:

$$\text{frequency} = \frac{1}{2\pi}\sqrt{\frac{3EI}{mL^3}}$$

The frequency of the cantilever is a function of E, I, m and L. So, in functional notation we can write:

$$\text{frequency} = f(E, I, m, L)$$

Example

If we have y as a function of x and described by the relationship $y = x^2$, what are the values of (a) $f(0)$, (b) $f(2)$?

(a) The function is described by $y = f(x) = x^2$. Thus $f(0)$ is the value of the function when $x = 0$ and so is 0.
(b) $f(2)$ is the value of the function when $x = 2$ and so is 4.

Example

Determine the values of (a) $f(2)$, (b) $f(4)$ if we have y as a function of x and defined by:

$y = 1$ for $0 \le x < 3$, $y = 2(x - 3) + 1$ for $3 \le x$

(a) The value of the function at $x = 2$ is given by the first relationship as 1.
(b) The value of the function at $x = 4$ is given by the second relationship as $y = 2(4 - 3) + 1 = 3$.

1.1.1 Combinations of functions

Many of the functions encountered in engineering and science can be considered to be combinations of other functions. Suppose we have the function $y = f(x) = x^2 + 2x$. We can think of the function $f(x)$ as resulting from the combination of two functions $g(x)$ and $h(x)$. One of the functions takes an input of x and gives an output of x^2 and the other takes an input of x and gives an output of $2x$.

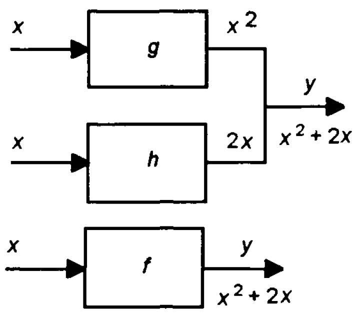


Figure 1.6 *Combination of functions g(x) and h(x)*

The two outputs are then added and we have $f(x) = g(x) + h(x)$. Figure 1.6 illustrates this.

Another way we can combine functions is by applying them in sequence. For example, if we have $h(x) = 2x$ and $g(x) = x^2$, then suppose we have the arrangement shown in Figure 1.7. The input of x to the g function box results in an output of x^2. The h function box takes its input and doubles it. Thus for an input of x^2 we have an output of $2x^2$, thus $f(x) = h\{g(x)\} = 2x^2$. Note that the order of the function boxes is important.

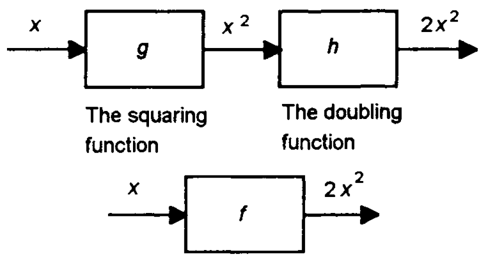


Figure 1.7 *Combination of functions to give the function f(x)*

Problems 1.1

1 If we have y as a function of x and defined by the equation $y = 2x + 3$, what are the values of (a) $f(0)$, (b) $f(1)$?

2 If we have y as a function of x and defined by the equation $y = x^2 + x$, what are the values of (a) $f(0)$, (b) $f(2)$?

3 If y is a function of x defined by the following equations, find the values of $f(0)$ and $f(1)$.

(a) $y = x^2 + 3$, (b) $y = x + 4$, (c) $y = (x + 1)^2 - 3$

4 Determine the values of (a) $f(0.5)$, (b) $f(2)$ if we have y as a function of x and defined by:

$y = 2$ for $0 \leq x < 1$, $y = 1$ for $1 \leq x$

5 Determine the values of (a) $f(1)$, (b) $f(3)$ if we have y as a function of t and defined by:

$y = 0$ for $0 \leq t < 2$, $y = 2(t - 2)$ for $2 \leq t$

6 The voltage in an electrical circuit is supplied by a constant voltage source of 10 V. If the voltage is switched on after time $t = 2$ s, state the equations defining the step voltage at any time t.

7 Sketch the periodic waveform described by the following equations:

$y = f(t) = t$ for $0 \leq t < 2$ and $y = f(t) = 2 - t$ for $2 \leq t < 4$

8 The period of oscillation T of a simple pendulum is a function of the length L of the pendulum, being defined by the equation

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where g is the acceleration due to gravity. What are the values of (a) $f(1)$, (b) $f(10)$ if g can be taken as 10 m/s^2?

9 The velocity v in metres per second of a moving object is a function of the time t in seconds, being defined by $v = 2 + 5t$. What are the values of (a) $f(0)$, (b) $f(1)$?

10 If $g(x) = 2x$ and $h(x) = x + 1$, what are (a) $g(x) + h(x)$, (b) $g\{h(x)\}$, (c) $h\{g(x)\}$?

11 If $f(x) = x^2 + 1$, $g(x) = 3x$ and $h(x) = 3x + 2$, determine: (a) $f(x) + g(x)$, (b) $f\{g(x)\}$, (c) $g\{f(x)\}$, (d) $f(x) - h(x)$, (e) $f\{h(x)\}$.

1.2 Linear functions

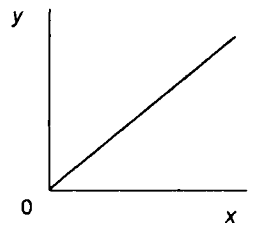


Figure 1.8 *Straight line graph with y directly proportional to x; the straight line thus passes though the origin*

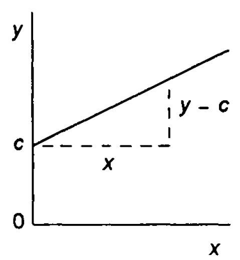


Figure 1.9 *Straight line graph not passing through the origin*

> **Key points**
>
> Linear functions are ones that provide a linear or straight line relationship between two variables when plotted as a conventional graph of one variable against the other. They can be defined by the general equation $y = mx + c$, where m is the gradient and c the intercept with the y axis.

This section is about a form of functions that is very commonly encountered in engineering, namely linear functions. Quite simply, linear functions are ones that provide a linear or straight line relationship between two variables when plotted as a conventional graph of one variable against the other.

The potential difference V across a resistor is a function of the current I through it. If the resistor obeys Ohm's law then $V = RI$, the potential difference is proportional to the current. If the current is doubled then the potential difference is doubled, if the current is trebled the potential difference is trebled. This means that a graph of V plotted against I is a straight line graph passing through the origin. *Gradient* is defined as the change in y value divided by the change in x value. Thus, for all straight line graphs passing through the origin (Figure 1.8), the gradient is constant and given by gradient $m = y/x$. Hence the equation of such a straight line is of the form:

$$y = mx \qquad [5]$$

where m is the gradient of the line. *Only* when we have such a relationship is y directly proportional to x.

Straight line graphs which do not pass through the origin (Figure 1.9) have a gradient, change in y value divided by change in x value, given by $m = (y - c)/x$, where c is the value of y when $x = 0$, i.e. the intercept of the straight line with the y-axis. Thus, such lines have the equation:

$$y = mx + c \qquad [6]$$

This is the equation which defines a straight line and is termed a *linear equation*. It is important to realise that with $c \neq 0$ that y is *not proportional* to x.

The gradient m of a straight line graph may be positive or negative. The gradient may also have a value of zero and this is a

line parallel to the x-axis. The intercept c may be positive or negative, or zero.

Example

State the gradients and intercepts of the graphs of the following equations: (a) $y = 2x + 3$, (b) $y = 2 - x$, (c) $y = x - 2$.

(a) This has a gradient of +2 and an intercept with the y-axis of +3. A positive gradient means that y increases as x increases.
(b) This has a gradient of −1 and an intercept with the y-axis of +2. A negative gradient means that y decreases as x increases.
(c) This has a gradient of +1 and an intercept with the y-axis of −2.

Example

During a test to find how the power of a CNC lathe varied with depth of cut, the following results were obtained:

Depth of cut d (mm)	0.51	1.02	1.52	2.03	2 54	3.00
Power P (W)	0.89	1.04	1.14	1.32	1.43	1.55

Use a graph to show that the function connecting the quantities d and P is of the form $y = mx + c$. Use this function to calculate the depth of cut when the power is 1 W.

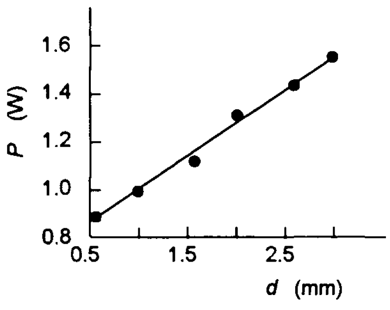


Figure 1.10 *Graph of power–depth of cut*

Figure 1.10 shows the graph with P on the y-axis and d on the x-axis. The graph line represents the line of best fit through all the points and may therefore be prone to some error. Because it is a straight line, the function is of the form $y = mx + c$ and so we have:

$$P = md + c$$

The slope m of the graph is about 0.27 and the point where the line would intercept the P axis when $d = 0$ is about 0.76. The function is thus:

$$P = 0.27d + 0.76$$

We can check the integrity of the above equation by substituting values from the table of observed results, say $d = 2.03$ mm. This gives:

$$P = 0.27 \times 2.03 + 0.76 = 1.31$$

If we now refer back to the table of results we see that this is close to what is given there, i.e. 1.32 W.

To finally answer the question regarding the depth of cut to be expected when the power is 1 W:

$1 = 0.27d + 0.76$

Hence d is about 0.89 mm.

Example

A pressure measurement system using a piezoelectric transducer is set up as represented by the system diagram of Figure 1.11. As the input pressure signal is altered, corresponding output readings are taken off the oscilloscope screen and the results shown below were obtained:

Pressure (bar)	0.5	1.0	1.5	2.0	2.5	3.0
Output voltage (mV)	32	50	90	125	160	200

Pressure (bar)	3.5	4.0	4.5	5.0	5.5
Output voltage (mV)	220	260	280	320	350

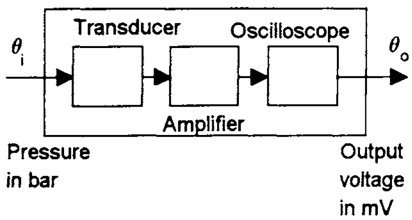


Figure 1.11 *Pressure measurement system*

Use a graph to show that the function connecting the quantities output voltage θ_o to the pressure θ_i is of the form $y = mx + c$. The static sensitivity of such a measurement system may be defined as the change in output signal divided by the change in the corresponding input signal. Determine the static sensitivity.

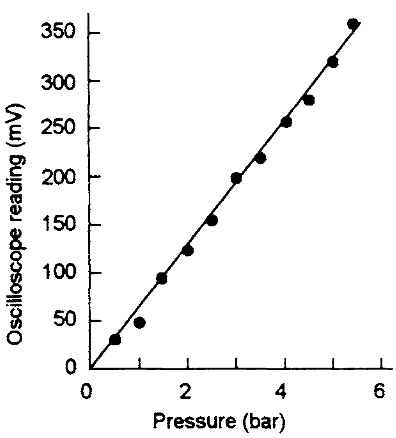


Figure 1.12 *Graph of output voltage–input pressure for pressure measurement system*

Figure 1.12 shows the graph obtained by plotting the above values. The graph line represents the line of best fit through all the points and may therefore be prone to some error. Because it is a straight line, the function is of the form $y = mx + c$ and so we have:

$\theta_o = m\theta_i + c$

From the graph, the approximate slope m is 350/5.5 = 63.6 mV/bar. The line passes through the origin and so c = 0. Hence:

$\theta_o = 63.6\theta_i$

The static sensitivity K is:

$$K = \frac{\Delta\theta_o}{\Delta\theta_i}$$

The symbol Δ in front of a quantity is used to indicate an increment of that quantity. But this is just the slope of the graph and so the static sensitivity is 63.6 mV/bar.

Problems 1.2

1 State which of the following will give a straight line graph and, if so, whether it passes through the origin:
(a) A graph of the extension of a spring plotted against the applied load when the extension is proportional to the applied load.
(b) A graph of the resistance R of a length of resistance wire plotted against the temperature t when $R = R_0(1 + \alpha t)$, with R_0 and α being constants.
(c) A graph of the distance d travelled by a car plotted against time t when $d = 10 + 4t^2$.
(d) A graph is plotted of the pressure p of a gas against its volume v, the pressure being related to the volume by Boyle's law, i.e. pv = a constant.

2 Determine the straight line equations for the following data if linear functional relationships can be assumed:
(a) The current i and time t over a period of time if at the beginning of the time we have i = 2 A and t = 0 s and at the end we have i = 3 A and t = 2 s.
(b) The extension e of a strip of material as a function of its length L when subject to constant stress, given that:

e in mm	0.60	0.72	0.84	0.96
L in m	0.5	0.6	0.7	0.8

1.3 Quadratic functions

A *linear function* is one where the equation defining the function is of the form $y = mx + c$. The highest power of a variable is 1. This is only one type of function. Here we look at another form, the quadratic function, and examine its defining equation.

The term *quadratic function* is used for a function $y = f(x)$ where the defining equation has the general form:

$$f(x) = ax^2 + bx + c \qquad [7]$$

where a, b and c are constants. The highest power of the variable is 2.

Key point

Quadratic functions have defining equations in which the highest power of the variable is 2.

Quadratic equations occur often in engineering. An example of such an equation in engineering occurs with the e.m.f. E of a thermocouple which can often be described by:

$$E = at + bt^2$$

where t is the temperature and a and b are constants. Other examples occur in the relationships for the bending moment M for bending beams, such as that for a cantilever propped at its free end:

$$M = \tfrac{1}{2}wx^2 - \tfrac{5}{8}wLx + \tfrac{1}{8}wL^2$$

where x is the distance from the free end of a cantilever of length L and w the distributed load per unit length.

The linear equation and the quadratic equation are just two examples of what are termed *polynomials*. A polynomial is the term used for any equation involving powers of the variable which are positive integers. Such powers can be 1, 2, 3, 4, 5, etc. For example, $x^4 + 4x^3 + 2x^2 + 5x + 2 = 0$ is a polynomial with the highest power being 4.

1.3.1 Factors and roots

To factorise a number means to write it as the product of smaller numbers. Thus, for example, we can factor 12 to give $12 = 3 \times 4$. The 3 and 4 are factors of 12. If the 3 and the 4 are multiplied together then 12 is obtained. To *factorise a polynomial* means to write it as the product of simpler polynomials. Thus for the quadratic expression $x^2 + 5x + 6$ we can write:

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

$(x + 2)$ and $(x + 3)$ are factors. If the two factors are multiplied together then the $x^2 + 5x + 6$ is obtained. Note that, in general:

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd \qquad [8]$$

If we have $u \times v = 0$ then we must have either $u = 0$ or $v = 0$ or both u and v are 0. This is because 0 times any number is 0. Thus if we have the quadratic equation $x^2 + 5x + 6 = 0$ and rewrite it as $(x + 2)(x + 3) = 0$, then we must have either $x + 2 = 0$, or $x + 3 = 0$ or both equal to 0. This means that the solutions to the quadratic equation are the solutions of these two linear equations, i.e. $x = -2$ and $x = -3$. These values are called the *roots* of the equation. We can check these values by substituting them into the quadratic equation. Thus for $x = -2$ we have $4 - 10 + 6 = 0$ and thus $0 = 0$. For $x = -3$ we have $9 - 15 + 6 = 0$ and thus $0 = 0$.

Key point

We can solve a quadratic equation by:

1 Factoring the quadratic.

2 Setting each factor equal to 0.

3 Solving the resulting linear equations.

Example

Factorise and hence solve the quadratic equation $x^2 - 3x + 2 = 0$.

To factorise this equation we need to find the two numbers which when multiplied together will give 2 and which when added together will give -3.

If we multiply −1 and −2 we obtain 2 and the addition of −1 and −2 gives −3. Thus we can write:

$(x - 1)(x - 2) = 0$

The solutions are thus given by $x - 1 = 0$, i.e. $x = 1$, and $x - 2 = 0$, i.e. $x = 2$.

We can check these values by substituting them into the original equation, $x^2 - 3x + 2$. Thus, for $x = 1$ we have $1 - 3 + 2 = 0$ and so $0 = 0$. For $x = 2$ we have $4 - 6 + 2 = 0$ and so $0 = 0$.

Completing the square

Consider the equation $x^2 + 6x + 9 = 0$. This equation can be factorised to give $(x + 3)(x + 3) = 0$, i.e. $(x + 3)^2 = 0$. It is a perfect square, both the roots being the same. Now consider the equation $x^2 + 6x + 2 = 0$. What are the factors? We can rewrite the equation as:

$x^2 + 6x = -2$

If we add 9 to both sides of the equation then we obtain

$x^2 + 6x + 9 = -2 + 9 = 7$

The left-hand side of the equation has been made into a perfect square by the adding of the 9. Thus we can write:

$(x + 3)^2 = 7$

This means that $x + 3$ must be one of the square roots of 7, i.e.

$x + 3 = \pm\sqrt{7}$

The plus or minus is because every positive number has two square roots, one positive and one negative. Thus we have $(+\sqrt{7}) \times (+\sqrt{7}) = 7$ and $(-\sqrt{7}) \times (-\sqrt{7}) = 7$. Hence:

$x = -3 \pm \sqrt{7}$

The two solutions are thus $x = -3 + \sqrt{7}$ and $x = -3 - \sqrt{7}$.

This method of determining the roots of a quadratic equation is known as *completing the square*. In the above discussion the left-hand side of the equation was made into a perfect square by the adding of 9. How do we determine what number to add in order to make a perfect square? Any expression of the form $x^2 + ax$ becomes a perfect square when we add $(a/2)^2$, since:

Key point

The procedure for determining the roots of a quadratic equation by completing the square can be summarised as:

1 Put the equation in the form $x^2 + ax = b$.

2 Determine the value of $(a/2)$.

3 Add $(a/2)^2$ to both sides of the equation to give:

$x^2 + ax + (a/2)^2 = b + (a/2)^2$

4 Hence obtain the equation:

$(x + a/2)^2 = b + (a/2)^2$

5 Determine the two roots by taking the square root of both sides of the equation, i.e.

$$x + \frac{a}{2} = \pm\sqrt{b + \left(\frac{a}{2}\right)^2}$$

$$x^2 + ax + \left(\frac{a}{2}\right)^2 = \left(x + \frac{a}{2}\right)^2 \qquad [9]$$

Thus for $x^2 + 6x$ we have $a = 6$ and so $(a/2) = 3$; we add $3^2 = 9$.

The above rule for completing the square only works if the coefficient of x^2, i.e. the number in front of x^2, is 1. However, if this is not the case we can simply divide throughout by that coefficient in order to make it 1.

Example

Use the method of completing the square to solve the quadratic equation $x^2 + 10x - 4 = 0$.

The quadratic equation can be written as:

$x^2 + 10x = 4$.

Adding $(10/2)^2 = 25$, to both sides of the equation gives:

$x^2 + 10x + 25 = 4 + 25$

Thus:

$(x + 5)^2 = 29$

Hence, $x + 5 = \pm\sqrt{29} = \pm 5.39$ and the solutions of the quadratic equation are $x = +5.39 - 5 = 0.39$ and $x = -5.39 - 5 = -10.39$.

We can check these values by substituting them in the equation $x^2 + 10x - 4 = 0$. Thus, for $x = 0.39$ we have $0.39^2 + 3.9 - 4 = 0.05$, which, because of the rounding used to limit the number of decimal places in determining the root, is effectively zero. For the other solution of $x = -10.39$ we have $(-10.39)^2 + 103.9 - 4 = 0.05$, which, because of the rounding used to limit the number of decimal places in determining the root, is effectively zero.

The quadratic formula

Consider the quadratic equation $ax^2 + bx + c = 0$. To obtain the roots by completing the square method, we divide throughout by a to give:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

This can be written as:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

To make the left-hand side of the equation a perfect square we must add $(b/2a)^2$ to both sides of the equation. Hence:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

and so:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

Taking the square root of both sides of the equation gives

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Thus:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

and so we have the *general formula for the solution of a quadratic equation*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad [10]$$

Key point

The general formula for the solution of a quadratic equations is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad [10]$$

Consider the following three situations:

- If we have $(b^2 - 4ac) > 0$, then the square root is of a positive number. There are then *two distinct roots* which are said to be *real*.
- If we have $(b^2 - 4ac) = 0$, then the square root is zero and the formula gives just one value for x. Since a quadratic equation must have two roots, we say that the equation has *two coincident real roots*.
- If we have $(b^2 - 4ac) < 0$, then the square root is of a negative number. A new type of number has to be invented to enable such expressions to be solved. The number is referred to as a *complex number* and the roots are said to be *imaginary* (the roots in 1 and 2 above are said to be *real*). Such numbers are discussed later in this book.

Example

Determine, if they exist as real roots, the roots of the following quadratic equations:

(a) $4x^2 - 7x + 3$, (b) $x^2 - 4x + 4$, (c) $x^2 + 2x + 4$.

(a) Using the general quadratic formula [10], here we have $a = 4$, $b = -7$ and $c = 3$. Therefore:

$$x = \frac{+7 \pm \sqrt{49-48}}{8} = \frac{7 \pm 1}{8}$$

Therefore, $x = 1$ or $x = 0.75$ defines the two roots of the equation. We can now represent the function as:

$(x - 1)(x - 0.75)$

and this, when multiplied out, gives $x^2 - 1.75x + 0.76$ and which when multiplied by 4 gives the original equation $4x^2 - 7x + 3$.

(b) Using the general quadratic formula [10] gives:

$$x = \frac{-2 \pm \sqrt{16-16}}{2} = 2$$

Therefore, we have two roots with $x = 2$. We may now rewrite the equation in the form $(x - 2)(x - 2)$. This, when multiplied out, gives $x^2 - 4x + 4$.

(c) Using the general quadratic formula [10] gives:

$$x = \frac{-2 \pm \sqrt{4-16}}{2}$$

Since the term inside the square root is negative, we have *no real roots.*

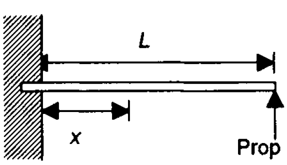


Figure 1.13 *Propped cantilever*

Example

Figure 1.13 shows a simple cantilever of length L, propped at its free end. It can be shown that the bending moment M of this type of cantilever is a function of the distance x measured from the fixed end of the beam, thus $M = f(x)$. The defining equation for the function is:

$$M = f(x) = \frac{Wx^2}{2} - \frac{5WLx}{8} + \frac{WL^2}{8}$$

where W is the distributed load in newtons per unit length. Using this quadratic formula, determine the positions along the beam at which the bending moment is zero (in engineering called the *points of contraflexure*).

When $M = 0$, we have:

$$\frac{Wx^2}{2} - \frac{5WLx}{8} + \frac{WL^2}{8} = 0$$

We can solve this by using the general quadratic formula [10]. Firstly, we can simplify the expression by multiplying through by 8 and taking the W term out as a factor:

$$4Wx^2 - 5WLx + WL^2 = W(4x^2 - 5Lx + L^2) = 0$$

and so the equation becomes $4x^2 - 5Lx + L^2 = 0$. Thus:

$$x = \frac{5L \pm \sqrt{(-5L)^2 - 4(4)(L^2)}}{2(4)}$$

$$x = \frac{5L \pm \sqrt{25L^2 - 16L^2}}{8}$$

Hence $x = L/4$ or $x = L$. The bending moment is thus zero at two locations (it has two points of contraflexure), i.e. at $L/4$ from the fixed end or at the extreme right-hand end of the beam when $x = L$.

Example

The distance s in metres moved by a vehicle over a period of time t seconds is defined by the equation $s = ut + \frac{1}{2}at^2$, with a being the constant acceleration and u the initial velocity. Assuming the vehicle commences motion with an initial velocity of 5 m/s and covers 84 m with a constant acceleration of 2 m/s^2, calculate the time over which this occurs.

Substituting the values in the equation gives:

$$84 = 5t + \tfrac{1}{2}(2)t^2$$

$$84 = 5t + t^2$$

Writing the equation in the general format:

$$t^2 + 5t - 84 = 0$$

Thus:

$$t = \frac{-5 \pm \sqrt{25 - 4(84)}}{2}$$

and so $t = -12$ s or $t = 7$ s. Since we cannot have a negative time, the only acceptable answer is $t = 7$ s.

The solution may be checked by substituting into the original equation $t^2 + 5t - 84 = 0$, when $t = 7$ s we have $7^2 + 5(7) - 84 = 0$. Since this is true, our solution holds.

Example

The total surface area A of a cylinder of radius r and height h is given by the equation $A = 2\pi r^2 + 2\pi rh$. If $h = 6$ cm, what will be the radius required to give a surface area of 88/7 cm^2? Take π as 22/7.

Putting the numbers in the equation gives

$$\frac{88}{7} = 2\times\frac{22}{7}r^2 + 2\times\frac{22}{7}\times 6r$$

Multiplying throughout by 7 and dividing by 44 gives

$$2 = r^2 + 6r$$

Hence we can write

$$r^2 + 6r - 2 = 0$$

and so:

$$r = \frac{-6\pm\sqrt{36-4\times 1\times(-2)}}{2\times 1} = \frac{-6\pm\sqrt{44}}{2} = \frac{-6\pm 6.63}{2}$$

Hence the solutions are $r = -6.32$ cm and $r = 0.32$ cm. The negative solution has no physical significance. Hence the solution is a radius of 0.32 cm.

We can check this value of 0.32 cm by substitution in the equation $2 = r^2 + 6r$. Hence $0.10 + 1.92 = 2.02$, which is effectively 2 bearing in mind the rounding of the root value to two decimal places that has occurred.

Problems 1.3

1 Determine, if they exist, the real roots of the following quadratic functions:

(a) $x^2 + 2x - 4$, (b) $x^2 + 3x + 1$, (c) $x^2 - 2x - 1$, (d) $x^2 + x + 2$.

2 The e.m.f. E of a thermocouple is a function of the temperature T, being given by $E = -0.02T^2 + 6T$. The e.m.f. is in μV and the temperature in °C. Determine the temperatures at which the e.m.f. will be 200 μV.

3 When a ball is thrown vertically upwards with an initial velocity u from an initial height h_0, the height h of the ball is a function of the time t, being given by $h - h_0 = ut - 4.9t^2$. Determine the times for which the height is 1 m, if $u = 4$ m/s and $h_0 = 0.5$ m.

4 The deflection y of a simply supported beam of length L when subject to an impact load of mg dropped from a height h on its centre is obtained by equating the total energy released by the falling load with the strain energy acquired, i.e.

$$mgh + mgy = \frac{24EI}{L^3}$$

Hence obtain an expression for the deflection y.

5 The height h risen by an object, after a time t, when thrown vertically upwards with an initial velocity u is given by the equation $h = ut - \frac{1}{2}gt^2$, where g is the acceleration due to gravity. Solve the quadratic equation for t if $u = 100$ m/s, $h = 150$ m and $g = 9.81$ m/s^2.

6 A rectangle has one side 3 cm longer than the other. What will be the dimensions of the rectangle if the diagonals have to have lengths of 10 cm? Hint: let one of the sides have a length x, then the other side has a length of 3 + x. The Pythagoras theorem can then be used.

1.4 Inverse functions

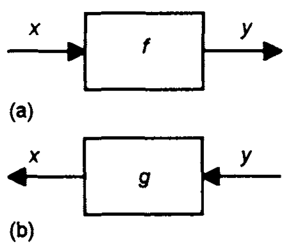


Figure 1.14 *(a) y = f(x), (b) x = g(y)*

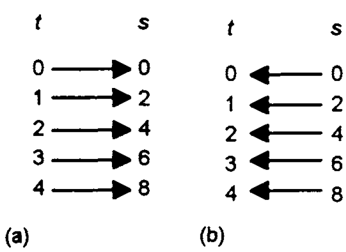


Figure 1.15 *For (a) function s = 2t, (b) function t = s/2*

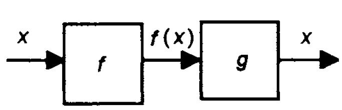


Figure 1.16 *x = g{f(x)}*

So far, in the treatment of a function we have started with a value of the independent variable x and used the function to find the corresponding value of the dependent variable y (Figure 1.14(a)). However, suppose we are given a value for y and want to find x (Figure 1.14(b)). For example, we might have distance s as a function of time t, e.g. $s = 2t$. Given a value of the independent variable t we can use the function to determine s. Suppose though that we are given a value of the dependent variable s and have to determine the corresponding t value? With the given equation we can rearrange it to give $t = s/2$. The function from t to s is $f(t)$, the function from s to t is a different function $g(s)$.

Figure 1.15(a) shows some values for the $s = f(t)$ function described by the equation $s = 2t$. Figure 1.15(b) shows the function obtained by reversing the arrows, i.e. starting with time values deducing the corresponding distance values. This figure represents the *inverse* relationship.

Note that there is a simple point of significance here: if we use $s = 2t$ to calculate a value for s given a value of t and then use the inverse by taking that value of t to calculate a value of s, we end up back where we started with our original value of s. This leads to a method of specifying an inverse function. Consider the arrangement shown in Figure 1.16. Here the g function system box operates on the output from the f function box in order to undo the work of the f box. Because the g function is undoing the work of the f function it is said to be the *inverse* of f. We may, therefore, write:

$$g\{f(x)\} = x \qquad [11]$$

This equation [11] is used to define an inverse function:

Key point

If f is a function of x then the function g which satisfies g{f(x)} = x for all values of x in the domain of f is called the inverse of f.

If f is a function of x then the function g which satisfies g{f(x)} = x for all values of x in the domain of f is called the inverse of f.

With regard to notation: the inverse of a function f of x is written as $f^{-1}(x)$. Note that $f^{-1}(x)$ does *not* mean $1/f(x)$, it is simply the notation to indicate the inverse function (the -1 not indicating a a power -1!). $f^{-1}(x)$ takes an input which is some function of x and inverts it to give an output of x. Thus the above definition gives:

$$f^{-1}\{f(x)\} = x \qquad [12]$$

As an illustration, consider a function f which adds 2, i.e. we have $f(x) = x + 2$ (Figure 1.17(a)). Then the inverse is a function that subtracts 2 in order to undo the action of the f function (Figure 1.17(b)). Thus, $f^{-1}(x) = x - 2$ and so if we put into the function $x + 2$ we obtain $(x + 2) - 2$.

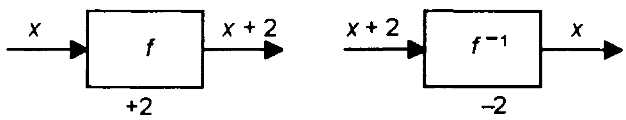


Figure 1.17 *(a) The function which adds 2, (b) the inverse function which subtracts 2*

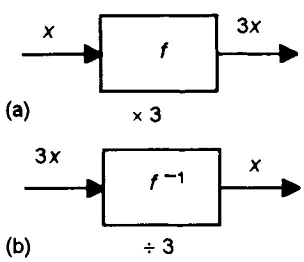


Figure 1.18 *(a) The function which multiplies by 3, (b) the inverse function which divides by 3*

As a further illustration, consider a function f which multiplies by 3, i.e. we have $f(x) = 3x$ (Figure 1.18(a)). Then the inverse is a function that divides by 3 in order to undo the action of the f function (Figure 1.18(b)). Thus $f^{-1}(x) = x/3$.

As another illustration, consider a function which multiplies by 3 and then adds 2, i.e. $f(x) = 3x + 2$. Here we have operated on the input x to the function twice, initially we multiplied the x by 3 and then we added the number 2 (Figure 1.19(a)). To arrive back at the original input, the inverse do two things (the reverse operation of those just detailed), namely initially subtract 2 and then secondly divided through by 3 (Figure 1.19(b)). Note that you must undo things in the reverse order to which they were done with the function f.

Key point

A function can only have an inverse if there is a one-to-one relationship between the input to a function and its output. Some functions have inverses for just some part of their domain. For example, the function $y = f(x) = x^2$ with an input of +1 or −1 gives $y = +1$. Thus if we take the inverse we do not know whether the result should be +1 or −1. unless we place some resriction on the domain, e.g. restrict it to just positive values of x.

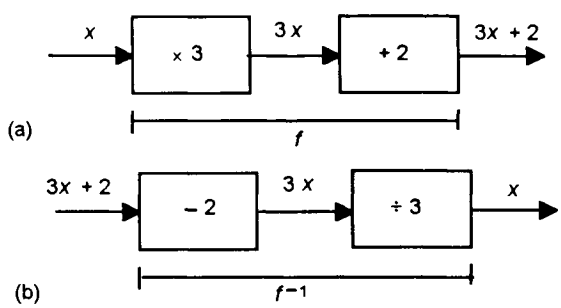


Figure 1.19 *(a) The functions which multiply by 3 and then add 2, (b) the inverse functions which subtract 2 and then divide by 3*

The above illustrations are rather basic functions. We will investigate more complex ones later. However, the basic rules still apply and, once understood, will provide a solid foundation from which to build more complex relationships.

Example

If $f(x) = 2x$, what is the inverse function?

The initial function $f(x) = 2x$ multiples by 2. Therefore, to reverse the process we simply divide by 2. Thus, $f^{-1}(x) = 1/2x$.

Example

If $f(x) = 2x + 3$, what is the inverse function?

$f(x) = 2x + 3$ involves doubling the input and then adding 3. The inverse is thus subtracting 3 from the input and then halving. Thus the inverse function is:

$$f^{-1}(x) = \frac{x-3}{2}$$

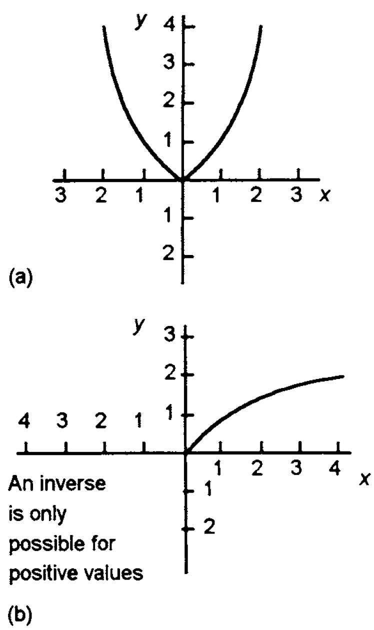


Figure 1.20 *(a) $y = f(x)$, (b) $y = f^{-1}(x)$*

1.4.1 Graphs of f and f^{-1}

We can use the above rules for a function and its inverse to find the graph of an inverse function from a graph of the function. Consider the graph of $y = f(x)$ shown in Figure 1.20(a). This is the graph described by the equation $y = x^2$. What is the graph of the inverse function $f^{-1}(x)$? This will be the graph of $y = \sqrt{x}$ (Figure 1.20(b)) since the function $\sqrt{x}$ is what we need to apply to undo the function x^2.

If we examine the two graphs we find that the inverse f^{-1} is just the reflection of the graph of f in the line $y = x$ (Figure 1.21). This is true for any function when it possesses an inverse.

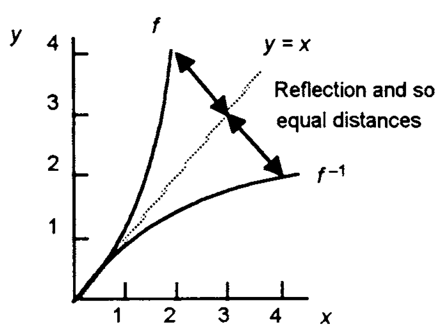


Figure 1.21 *The inverse as a reflection of function in line $y = x$*

Problems 1.4

1 Determine the inverses of:

(a) $f(x) = 5x - 3$, (b) $f(x) = 4 + x$, (c) $f(x) = x^3$, (d) $f(x) = 2x^3 - 1$.

2 Does the function $f(x) = x^2$, have an inverse for all real values of x?

3 For each of the following functions, restrict the domain so that there is an inverse and then determine it:

(a) $f(x) = (x - 1)^2$, (b) $f(x) = (x + 1)^2 - 4$.

1.5 Circular functions

This section focuses on the so-called circular or trigonometric functions. Such functions are widely used in engineering. Thus in describing oscillations, whether mechanical of perhaps a vibrating beam or electrical and alternating current, the equations used to define the quantity which fluctuates with time is likely to involve a trigonometric function.

As an illustration, consider the mechanical oscillation of a mass on the end of a spring when is just vibrates up-and-down when the mass is given a vertical displacement (Figure 1.22). With very little damping, the mass will oscillate up-and-down for quite some time. Figure 1.23 shows how the displacement varies with time.

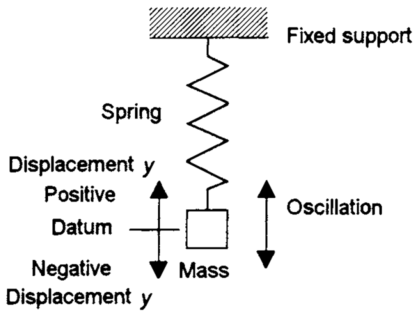


Figure 1.22 *Oscillation of a mass on the end of a vertical spring*

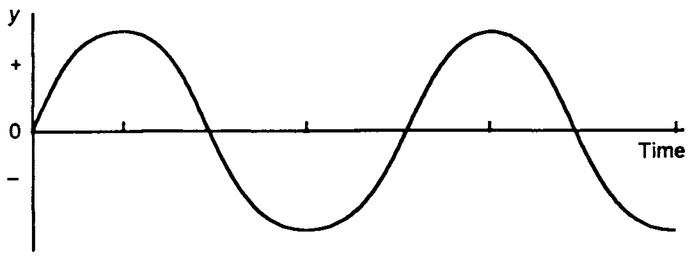


Figure 1.23 *Displacement y variation with time for the oscillating mass when damping is virtually absent*

If we look at the situation when there is noticeable damping present, then the displacement variation with time looks more like Figure 1.24. The difference between this graph and Figure 1.23 is that, though we have a similar form of graph, the effect of the damping is that the amplitude decreases with time.

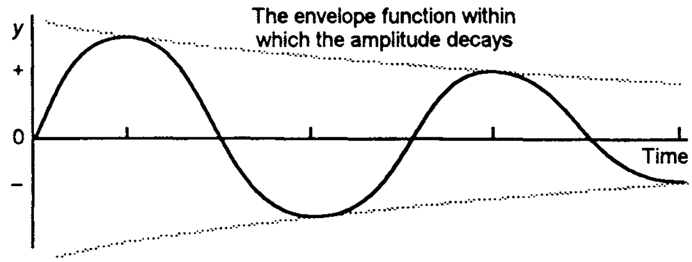


Figure 1.24 *Displacement y variation with time for the oscillating mass when damping is noticeable*

We can derive an equation to represent the variation of displacement with time in the absence of damping by using a simple model. Suppose we draw a circle with a radius OA equal to the amplitude of the oscillation, i.e. the maximum displacement, and consider a point P moving round the circle with a constant angular velocity ω (Figure 1.25) and starting from the horizontal. The vertical projection of the rotating radius OP gives a displacement–angle graph. Since the radius OP is rotating with a constant angular velocity, the angle rotated is proportional to the time. The result is a graph which replicates that of the undamped oscillating mass.

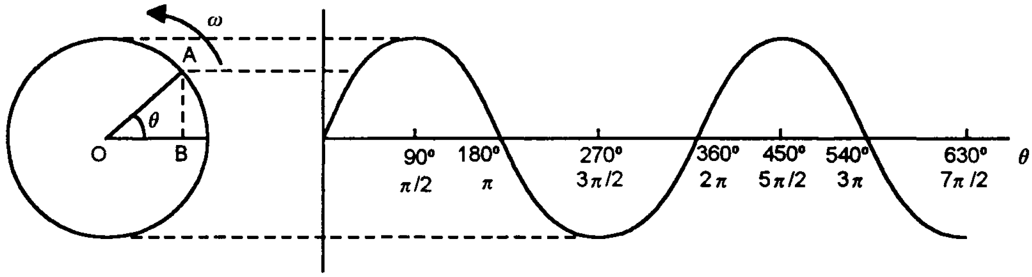


Figure 1.25 *The vertical projection AB of the rotating radius gives a displacement–time graph*

Key points

The convention used for angles is that they are referenced from zero degrees and when measured in an anticlockwise direction are termed positive angles. One unit for angles is degrees (Figure 1.26(a)). However, when angles occur in equations in engineering, it is usual to describe them in radians. One complete rotation of a radius is a rotation through 360°.

One radian is the angle swept so that the arc formed is the same length as the radius, hence since one complete sweep of a radius is an arc length of $2\pi r$, then one complete rotation is 2π radians (Figure 1.26(b)). Thus 360° = 2π radians (rad) and so 1 rad = $360/2\pi$ = 57.3°. Hence:

$$\theta_{radians} = \frac{\theta_{degrees}}{57.3}$$

$$\theta_{degrees} = \theta_{radians} \times 57.3$$

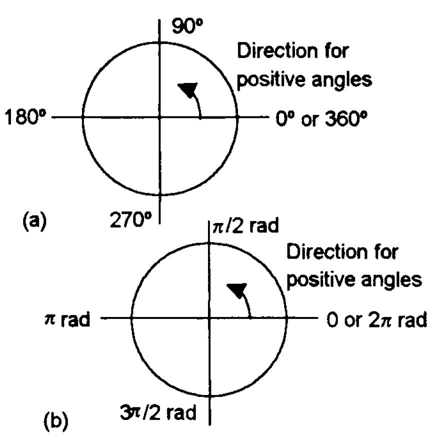


Figure 1.26 *Conventions for angles in degrees and radians*

Since AB/OA = sin θ we can write:

$$AB = OA \sin\theta$$

where AB is the vertical height of the line at some instant of time, OA being its length. The maximum value of AB will be OA and occur when θ = 90°. But a constant angular velocity ω means that in a time t the angle θ covered is ωt. Thus the vertical projection AB of the rotating line will vary with time and is described by the equation:

$$AB = OA \sin\omega t$$

If y is the displacement of the alternating mass and A the amplitude of its oscillation, the equation can be written as:

$$y = A \sin\omega t \qquad [13]$$

This type of oscillation is called *simple harmonic motion.*

It is usual to give angular velocities in units of radians per second, an angular rotation through 360° being a rotation through 2π radians. Since the periodic time T is the time taken for one cycle of a waveform, then T is the time taken for OA to complete one revolution, i.e. 2π radians. Thus:

$$T = \frac{2\pi}{\omega}$$

The frequency f is $1/T$ and so $\omega = 2\pi f$. Because ω is just 2π times the frequency, it is often called the *angular frequency.* The frequency f has units of hertz (Hz) or cycles per second and thus the angular frequency has units of s^{-1}. We can thus write the above equation as:

$$y = A \sin 2\pi ft \qquad [14]$$

We can use a similar model to describe alternating current; in this case the rotating radius OP is called a *phasor.* Thus the current i is related to its maximum value I by:

Key points

In $y = A \sin \omega t$, A is the amplitude, ω the angular frequency.
Angular frequency $\omega = 2\pi f$.
Period $T = 1/f$.

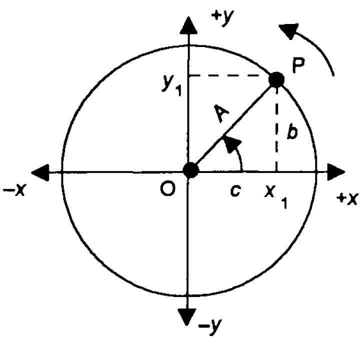


Figure 1.27 *Defining circular functions*

Key points

The definitions used to define the trigonometric ratios in terms of a right-angled triangle (Figure 1.28) are:

$\sin\theta = \frac{b}{c}$, $\cos\theta = \frac{a}{c}$, $\tan\theta = \frac{b}{a}$

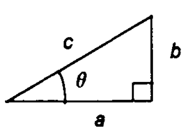


Figure 1.28 *Defining trig. ratios*

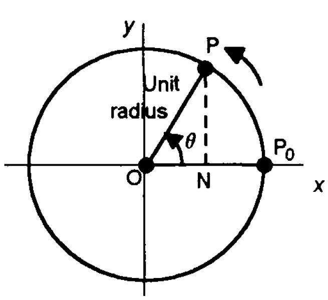


Figure 1.29 *Circular functions*

$i = I \sin \omega t$

For the damped oscillation, the amplitude decreases with time so we must figure out how to quantify this 'decay' and link it somehow with the basic sine wave function of the undamped system. As we will later discover, the damped oscillation is actually described by a combination of a sine function and an exponential function.

The circular functions

We can define the circular functions, i.e. the sine, cosine and tangent, in terms of the rotation of a radial arm of length A (it often represents the amplitude) in a circular path (Figure 1.27). Thus, we can define the sine of an angle as:

$$\sin\theta = \frac{b}{a}$$

But, with $b = y_1 - 0$, then $\sin\theta = y_1/A$ and so:

$$y_1 = A \sin\theta \qquad [15]$$

This relationship now enables us to calculate the vertical side of the triangle Ox_1P, or the y-coordinate of point P.

Likewise, we can define the cosine of an angle as:

$$\cos\theta = \frac{c}{A}$$

But, with $c = x_1 - 0$, then $\cos\theta = x_1/A$ and so:

$$x_1 = A \cos\theta \qquad [16]$$

This relationship now enables us to calculate the horizontal side of the triangle Ox_1P, or the x-coordinate of point P.

We can define the tangent of an angle in terms of the gradient of the line OP as:

$$\tan\theta = \frac{b}{c} = \frac{y_1}{x_1} \qquad [17]$$

Using equations [15] and [16] we can write equation [17] as:

$$\tan\theta = \frac{A\sin\theta}{A\cos\theta} = \frac{\sin\theta}{\cos\theta} \qquad [18]$$

With reference to Figure 1.27, as the point P moves around the circle, so the angle θ changes. The trigonometrical ratios can be defined in terms of the angles in a right-angled triangle. However, the above definitions allow us to define them for all angles, not just those which are 90° or less. Because they are defined in terms of a circle, they are termed *circular functions*.

Consider the motion of a point P around a unit radius circle (Figure 1.29). P_0 is the initial position of the point and P the position to which it has rotated. The radial arm OP in moving

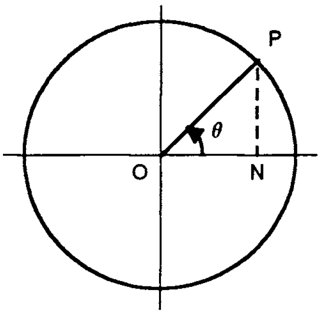


Figure 1.30 *First quadrant*

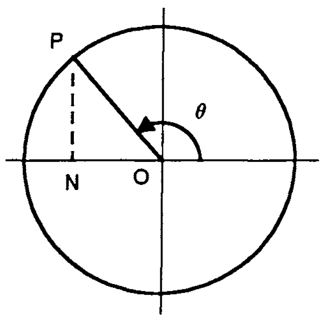


Figure 1.31 *Second quadrant*

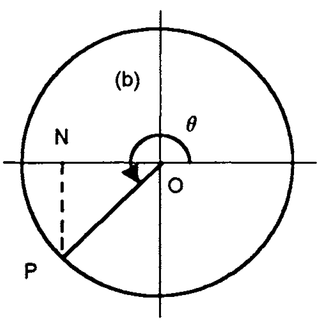


Figure 1.32 *Third quadrant*

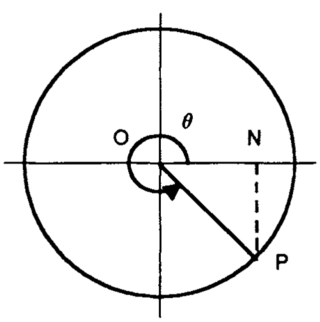


Figure 1.33 *Fourth quadrant*

from OP_0 has swept out an angle θ. The angle θ is measured between the radial arm and the OP_0 axis as a positive angle when the arm rotates in an anticlockwise direction. Since the circle has a unit radius, to obtain for angles up to 90° the same result as the trigonometric ratios defined in terms of the right-angled triangle, the perpendicular height NP defines the sine of the angle P_0OP and the horizontal distance ON defines the cosine of the angle P_0OP.

Consider the circular rotations for angles in each quadrant (note: the first quadrant is angles 0° to 90°, the second quadrant 90° to 180°, the third quadrant 180° to 270°, the fourth quadrant 270° to 360°):

1 *Angles between 0 and 90°*
When the radial arm OP is in the first quadrant (Figure 1.30) with $0 \leq \theta < \pi/2$, $0 \leq \theta < 90°$, both NP and ON are positive. Thus both the sine and the cosine of angle θ are positive. Since the tangent is NP/ON then the tangent of angle θ is positive. For example, sin 30° = +0.5, cos 30° = +0.87 and tan 30° = +0.58.

2 *Angles between 90° and 180°*
When the radial arm OP moves into the second quadrant (Figure 1.31) with $\pi/2 \leq \theta < \pi$, $90° \leq \theta < 180°$, NP is positive and ON negative. Thus the sine of angle θ is positive and the cosine negative. Since the tangent is NP/ON then the tangent of angle θ is negative. For example, sin 120° = 0.87, cos 120° = −0.50 and tan 120° = −1.73.

3 *Angles between 180° and 270°*
When the radial arm moves into the third quadrant (Figure 1.32) with $\pi \leq \theta < 3\pi/2$, $180° \leq \theta < 270°$, NP is negative and ON negative. Thus the sine of angle θ is negative and the cosine negative. Since the tangent is NP/ON then the tangent of angle θ is positive. For example, sin 210° = −0.5, cos 210° = −0.87 and tan 210° = +0.58.

4 *Angles between 270° and 360°*
When the radial arm is in the fourth quarter (Figure 1.33) with $3\pi/2 \leq \theta < 2\pi$, $270° \leq \theta < 360°$, NP is negative and ON positive. Thus the sine of angle θ is negative and the cosine positive. Since the tangent is NP/ON then the tangent of angle θ is negative. For example, sin 300° = −0.87, cos 300° = 0.5 and tan 300° = −1.73.

We can now summarise with Figure 1.34 as an aid to memory.

For angles greater than 2π rad (360°), the radial arm OP simply rotates more than one revolution. Negative angles are interpreted as a clockwise movement of the radial arm from OP_0.

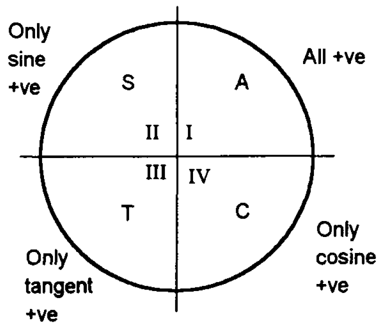


Figure 1.34 *Circular functions in the four quadrants*

Key point

Mathematicians call the sine function an 'odd' function. An *odd* function is defined as one which has:

$f(-x) = -f(x)$

An *even* function has:

$f(-x) = f(x)$

Cyclic functions

A cyclic function is one which repeats itself on a cyclic period. Thus, if we have a function $y = f(x)$ which is cyclic and repeats itself after a time T, then:

$$f(x) = f(x + T) = f(x + 2T) = \text{etc.} \qquad [19]$$

T is termed the *periodic time* and is the time taken to complete one cycle. Hence, if the frequency is f then f cycles are completed each second and so $T = 1/f$.

As the arm OP in Figure 1.35 rotates round-and-round its circular path, the values of its vertical projection NP is cyclic and generates the sine graph shown. Since the graph describes a periodic function of period 2π, then:

$$\sin\theta = \sin(\theta + 2\pi n) \qquad [20]$$

where $n = 0, \pm 1, \pm 2$, etc.

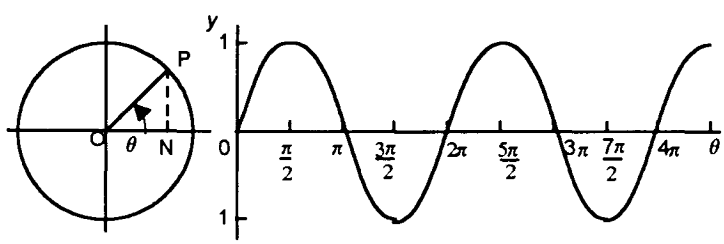


Figure 1.35 *Graph of* $y = \sin\theta$

Note that if OP rotates in a clockwise direction, i.e. the negative direction, then as θ is negative, this generates the sine function continued to the left of the origin O into the negative region (Figure 1.36). For negative values of θ, the sine function has the same values as the positive values except for a change in sign:

$\sin(-\theta) = -\sin\theta$

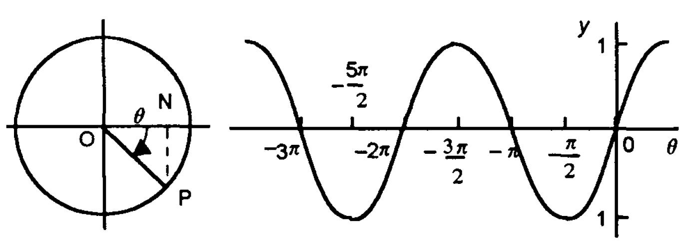


Figure 1.36 *Sine graph for negative angles*

To obtain the graph of $\cos\theta$ as the radial arm OP rotates round-and-round its circular path, we read off the values of its horizontal projection ON. Figure 1.37 shows the result. Since the graph describes a periodic function of period 2π, then:

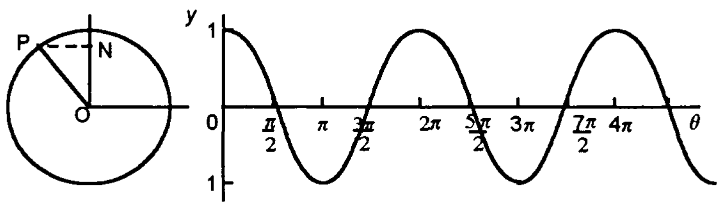


Figure 1.37 *Graph of y = cos θ*

$$\cos\theta = \cos(\theta + 2\pi n) \qquad [21]$$

where $n = 0, \pm 1, \pm 2$, etc.

Note that the graph of $y = \sin\theta$ is the same as that of $y = \cos\theta$ moved $\frac{1}{2}\pi$ to the right, while that of $y = \cos\theta$ is the same as $y = \sin\theta$ moved $\frac{1}{2}\pi$ to the left, i.e. $\sin\theta = \cos(\theta - \frac{1}{2}\pi)$ and $\cos\theta = \sin(\theta + \frac{1}{2}\pi)$.

In the projections of the radial arm OP to generate the sine or cosine graphs, we have let OP have the value of 1. If we consider a radial arm of length A, we have the same function but multiplied by A, i.e. $y = A\sin\theta$. The amplitude of the waveform is changed. To illustrate this look at the following functions and their graphs as plotted to the same scale and on the same axes (Figure 1.38): $y = 1\sin\theta$ with amplitude $A = 1$, $y = 4\sin\theta$ with amplitude $A = 4$ and $y = 0.5\sin\theta$ with amplitude $A = 0.5$.

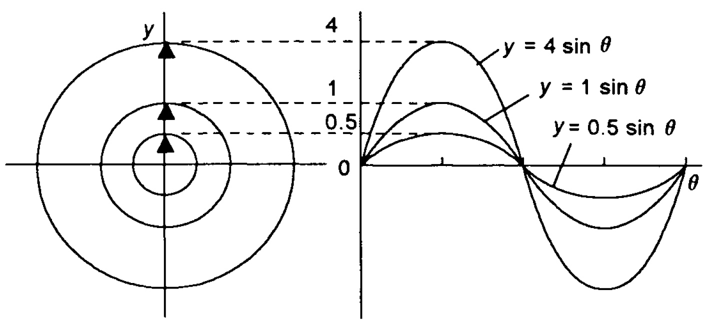


Figure 1.38 *Effect of changing A in y = A sin θ, only the amplitude of the graph waves is changed*

In engineering, we often encounter functions of the general form:

$$y = A\sin(\theta \pm \phi) \text{ or } y = A\cos(\theta \pm \phi) \qquad [22]$$

ϕ is the initial angle we start the rotating radial arm OP at and, as a consequence, ϕ is the angle by which the sine or cosine graph is moved to the left when positive and to the right when negative. It defines a phase shift of the complete waveform. Figure 1.39 illustrates this by showing the effect of a phase shift of $\pi/3$, i.e. 60°.

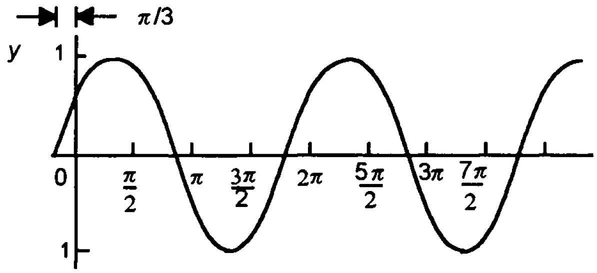


Figure 1.39 *Graph of y = sin (θ + π/3), showing the effect of a phase shift of π/3, i.e. 60°, as being to shift the graph to the left by that amount*

Now consider the graph of the function $y = \tan x$. For the radial arm OP rotating in a circle, the tangent is PN/OP (Figure 1.40). But if we draw a tangent to the circle at P_0 then, for a unit radius circle the tangent of the angle is P_0M. When the radius arm has moved to an angle between 90° and 180° then the tangent is P_0M_1. The graph describes a periodic function which repeats itself every period of π (not every 2π as for a sine or cosine function). Thus:

$$\tan \theta = \tan (\theta + \pi n) \qquad [23]$$

for $n = 0, \pm 1, \pm 2$, etc.

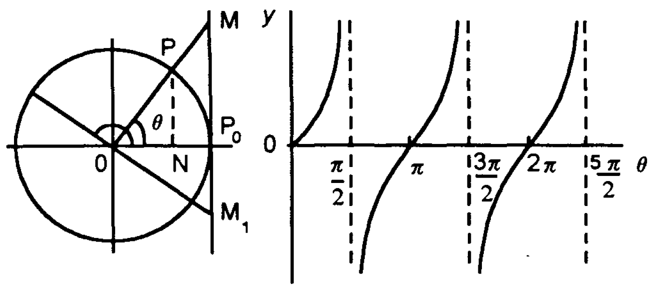


Figure 1.40 $y = \tan \theta$

Example

Draw graphs of $y = \cos \theta$ and $y = \cos 2\theta$ on the same axis and comment on how they differ.

A simple way to sketch the graphs is to formulate a table for values between $\theta = 0°$ and $\theta = 360°$ for $\cos \theta$ and $\cos 2\theta$, then plot the respective curves. So we have:

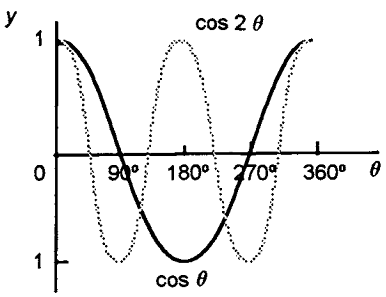


Figure 1.41 *Graphs of y = cos θ and y = cos 2 θ*

θ	0	30	45	60	90	120	135	150	180
2θ	0	60	90	120	180	240	270	300	360
$\cos\theta$	1.0	0.9	0.7	0.5	0	−0.5	−0.7	−0.9	−1.0
$\cos 2\theta$	1.0	0.5	0	−0.5	−1.0	−0.5	0	0.5	1.0

θ	210	225	240	270	300	315	330	360
2θ	420	450	480	540	600	630	660	720
$\cos\theta$	−0.9	−0.7	−0.5	0	0.5	0.7	0.9	1.0
$\cos 2\theta$	0.5	0	−0.5	−1.0	−0.5	0	0.5	1.0

Figure 1.41 shows the resulting graphs.

Example

Sketch the function y = 5 sin (θ + 30°) for values of θ between 0° and 360°.

The equation indicates that the waveform has an amplitude of 5 and a phase shift of +30°. Figure 1.42 shows the form of the function.

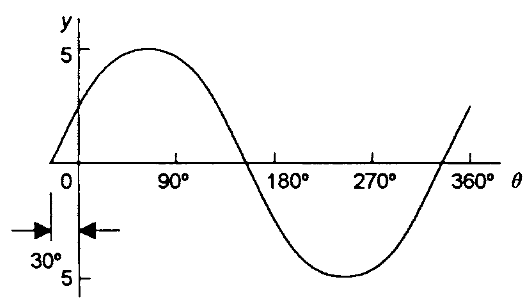


Figure 1.42 *Graph of y = 5 sin (θ + 30°)*

Maths in action

To illustrate a simple mechanical application, consider a piston head moving cyclically without damping, i.e. without friction, and represented by the spring–mass system shown in Figure 1.43. The spring represents the restoring or elastic driving force acting on the piston head.

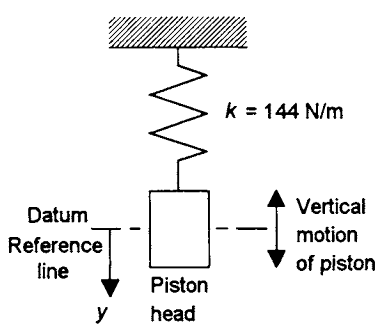


Figure 1.43 *Representation of a piston head*

We can find some basic properties of the system if we apply cyclic functions. We assume that the motion can be represented by the y-component of a rotating radial arm which rotates with a constant angular velocity and so is given by $y = A \sin\theta$, where y is the vertical displacement at a time t. Since $\theta = \omega t$, we can write:

$y = A \sin \omega t$

From mechanical theory, the angular frequency ω is:

$$\omega = \sqrt{\frac{k}{m}}$$

where k is the spring constant and m the mass. Hence $\omega = \sqrt{(144/4)} = 6$ rad/s and so $y = A \sin 6t$. We thus have y at a maximum when $\sin 6t = 1$, i.e. when $6t = 1.57$ and so $t = 0.26$ s.

If the maximum displacement, i.e. amplitude, of the piston is 0.1 m, we have $y = 0.1 \sin 6t$. Thus, at $t = 3$ s we have $y = 0.1 \sin 6(3) = 0.1 \sin 18 = 0.1$. Since 18 rad = 1031° then $y = 0.1(-0.75) = -0.075$ m. The minus sign indicates that the displacement is in the upward direction from the datum line.

Phasors

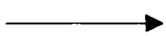


Figure 1.44 *Representing a phasor by an arrow-headed line*

A sinusoidal alternating current can be represented by the equation $i = I \sin \omega t$, where i is the current at time t and I the maximum current. In a similar way we can write for a sinusoidal alternating voltage $v = V \sin \omega t$, where v is the voltage at time t and V the maximum voltage. Thus we can think of an alternating current and voltage in terms of a model in which the instantaneous value of the current or voltage is represented by the vertical projection of a line rotating in an anticlockwise direction with a constant angular velocity. The term *phasor*, being an abbreviation of the term phase vector, is used for such rotating lines. The length of the phasor can represent the maximum value of the sinusoidal waveform (or the generally more convenient root-mean-square value, the maximum value is proportional to the root-mean-square value). The line representing a phasor is drawn with an arrowhead at the end that rotates and is drawn in its position at time $t = 0$, i.e. the phasor represents a frozen view of the rotating line at one instant of time of $t = 0$ (Figure 1.44).

Alternating currents or voltages which do not always start with zero values at time $t = 0$ and can be represented in general by:

$$i = I \sin(\omega t + \phi) \text{ or } v = V \sin(\omega t + \phi) \qquad [24]$$

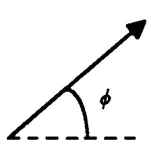


Figure 1.45 *Phasor with phase angle ϕ*

The phasor for such alternating currents or voltages is represented by a phasor (Figure 1.45) at an angle ϕ to the reference line, this line being generally taken as being the horizontal. The angle ϕ is termed the *phase angle*. We can describe such a phasor by merely stating its magnitude and phase angle (the term used is polar coordinates). Thus $2\angle 40°$ A describes a phasor with a magnitude, represented by its length, of 2 A and with a phase angle of 40°.

In discussing alternating current circuits we often have to consider the relationship between an alternating current through a

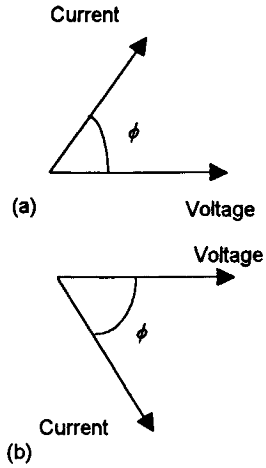


Figure 1.46 *(a) Current leading voltage, (b) current lagging voltage*

component and the alternating voltage across it. If we take the alternating voltage as the reference and consider it to be represented by a horizontal voltage phasor, then the current may have some value at that time and so be represented by another phasor at some angle ϕ. There is said to be a *phase difference* of ϕ between the current and the voltage. If ϕ has a positive value then the current is said to be *leading* the voltage, if a negative value then *lagging* the voltage (Figure 1.46).

Example

A sinusoidal voltage has a maximum value of 10 V and a frequency of 100 Hz. If the voltage has a phase angle of 30°, what will be the instantaneous voltage at times of (a) $t = 0$, (b) $t = 0.5$ ms?

The equation for the sinusoidal voltage will be:

$$v = V_m \sin (2\pi ft + \phi)$$

The term $2\pi ft$, i.e. ωt, is in radians. Thus, for consistency, we should express ϕ in radians. An angle of 30° is $\pi/6$ radians. Thus:

$$v = 10 \sin (2\pi \times 100t + \pi/6) \text{ volts}$$

It should be noted that it is quite common in engineering to mix the units of radians and degrees in such expressions. Thus you might see:

$$v = 10 \sin (2\pi \times 100t + 30°) \text{ volts}$$

However, when carrying out calculations involving the terms in the bracket there must be consistency of the units.

(a) When $t = 0$ then: $v = 10 \sin \pi/6 = 5$ V.

(b) When $t = 0.5$ ms then:

$$v = 10 \sin (2\pi \times 100 \times 0.5 \times 10^{-3} + \pi/6) \text{ volts}$$

and so $v = 10 \sin 0.838 = 7.43$ V.

1.5.2 Manipulating circular functions

Often in working through engineering problems, it is necessary to rearrange circular functions in a different format. This section looks at how we can do this.

Key points

The cosecant, secant and cotangent ratios are defined as the reciprocals of the sine, cosine and tangent:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

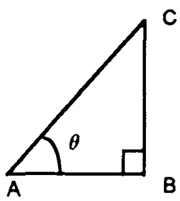


Figure 1.47 *Right-angled triangle*

Key points

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

$2\sin A\cos B = \sin(A + B) + \sin(A - B)$

$2\cos A\cos B = \cos(A + B) + \cos(A - B)$

$2\sin A\sin B = \cos(A - B) - \cos(A + B)$

$2\cos A\sin B = \sin(A + B) - \sin(A - B)$

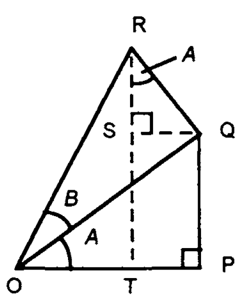


Figure 1.48 *Compound angle*

Trigonometric relationships

For the right-angled triangle shown in Figure 1.47, the *Pythagoras theorem* gives $AB^2 + BC^2 = AC^2$. Dividing both sides of the equation by AC^2 gives:

$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

Hence:

$$\cos^2\theta + \sin^2\theta = 1 \qquad [25]$$

Dividing this equation by $\cos^2\theta$ gives:

$$1 + \tan^2\theta = \sec^2\theta \qquad [26]$$

and dividing equation [25] by $\sin^2\theta$ gives:

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta \qquad [27]$$

Example

Simplify $\dfrac{\cos\theta}{1-\sin\theta} + \dfrac{\cos\theta}{1+\sin\theta}$.

$$\frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta} = \frac{\operatorname{cosec}\theta(1+\sin\theta) + \cos\theta(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{2\cos\theta}{1-\sin^2\theta} = \frac{2\cos\theta}{\cos^2\theta} = 2\sec\theta$$

Trigonometric ratios of sums of angles

It is often useful to express the trigonometric ratios of angles such as $A + B$ or $A - B$ in terms of the trigonometric ratios of A and B. In such situations the relationships shown in Key points prove useful.

As an illustration of how we can derive such relationships, consider the two right-angled triangles OPQ and OQR shown in Figure 1.48:

$$\sin(A+B) = \frac{TR}{OR} = \frac{TS + SR}{OR} = \frac{PQ + SR}{OR} = \frac{PQ}{OQ}\frac{OQ}{OR} + \frac{SR}{QR}\frac{Q}{O}$$

Hence:

$$\sin(A + B) = \sin A\cos B + \cos A\sin B \qquad [28]$$

If we replace B by $-B$ we obtain:

$$\sin(A - B) = \sin A\cos B - \cos A\sin B \qquad [29]$$

Key points

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - 2 \sin^2 A = 2 \cos^2 A - 1$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

If in equation [28] we replace A by $(\pi/2 - A)$ we obtain:

$$\cos (A + B) = \cos A \cos B - \sin A \sin B \qquad [30]$$

If in equation [29] we replace B by $-B$ we obtain:

$$\cos (A - B) = \cos A \cos B + \sin A \sin B \qquad [31]$$

We can obtain $\tan (A + B)$ by dividing $\sin (A + B)$ by $\cos (A + B)$ and likewise $\tan (A - B)$ by dividing $\sin (A - B)$ by $\cos (A - B)$. By adding or subtracting equations from above we obtain the relationships such as $2 \sin A \cos B$.

If, in the above relationships for the sums of angles A and B we let $B = A$ we obtain the double-angle equations shown in Key points.

Example

Solve the equation $\cos 2x + 3 \sin x = 2$.

Using equation [46] for $\cos 2x$ gives:

$$1 - 2 \sin^2 x + 3 \sin x = 2$$

This can be rearranged as:

$$2 \sin^2 x - 3 \sin x + 1 = 0$$

$$(2 \sin x - 1)(\sin x - 1) = 0$$

Hence $\sin x = ½$ or 1. For angles between 0° and 90°, $x =$ 30° or 90°.

Example

In an alternating current circuit, the instantaneous voltage v is given by $v = 5 \sin \omega t$ and the instantaneous current i by $i = 10 \sin (\omega t - \pi/6)$. Find an expression for the instantaneous power P at a time t given $P = vi$.

As $P = vi$ we have:

$$P = 5 \sin \omega t \{10 \sin (\omega t - \pi/6)\} = 50 \sin \omega t \{\sin (\omega t - \pi/6)\}$$

Using $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$ gives:

$$P = 25\{\cos (\pi/6) - \cos (2\omega t - \pi/6)\}$$

a cos *θ* + *b* sin *θ*

Sometimes it is useful to write an equation of the form $a \sin \theta + b \cos \theta$ in the form $R \sin (\theta - \alpha)$. We can do this by using the trigonometric formula for compound angles, e.g. equation [29] for $\sin (A - B)$. Thus:

$$R \sin (\theta - \alpha) = R(\sin \theta \cos \alpha + \cos \theta \sin \alpha)$$

Hence, we require:

$$R(\sin \theta \cos \alpha + \cos \theta \sin \alpha) = a \sin \theta + b \cos \theta$$

Therefore, comparing coefficients of the $\sin \theta$ terms:

$$R \cos \alpha = a$$

and comparing coefficients of the $\cos \theta$ terms:

$$R \sin \alpha = b$$

Dividing these two equations gives

$$\tan \alpha = \frac{a}{b} \qquad [32]$$

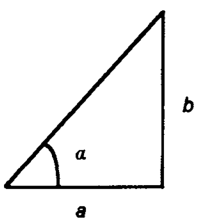


Figure 1.49 *Right-angled triangle*

This leads us to be able to describe the angle α by the right-angled triangle shown in Figure 1.49. Hence:

$$R = \sqrt{a^2 + b^2} \qquad [33]$$

Thus:

$$a \sin \theta + b \cos \theta = R \sin (\theta + \alpha) \qquad [34]$$

The Key points show other relationships which can be derived in a similar way.

Key points

$a \sin \theta + b \cos \theta = R \sin (\theta + \alpha)$

$a \sin \theta - b \cos \theta = R \sin (\theta - \alpha)$

with:

$R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \frac{a}{b}$

$a \cos \theta + b \sin \theta = R \cos (\theta - \alpha)$

$a \cos \theta - b \sin \theta = R \cos (\theta + \alpha)$

with:

$R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \frac{b}{a}$

Example

Express $3 \cos \theta + 4 \sin \theta$ in the form (a) $R \cos (\theta - \alpha)$, (b) $R \sin (\theta + \alpha)$.

(a) We can derive it by using the double-angle formula $\cos (A - B) = \cos A \cos B + \sin A \sin B$. Thus:

$$3 \cos \theta + 4 \sin \theta = R (\cos \theta \cos \alpha + \sin \theta \sin \alpha)$$

Thus $3 = R \cos \alpha$ and $4 = R \sin \alpha$. Hence tan a = 4/3 and so a = 53.1° or 0.93 rad. $R = \sqrt{(3^2 + 4^2)} = 5$. Hence:

$$3 \cos \theta + 4 \sin \theta = 5 \cos (\theta - 0.93)$$

(b) We can derive it by directly using the double-angle relationship sin ($A + B$) = sin A cos B + cos A sin B. Thus:

$3 \cos \theta + 4 \sin \theta = R(\sin \theta \cos a + \cos \theta \sin a)$

Thus $3 = R \sin a$ and $4 = R \cos a$. Hence tan a = 3/4 and so a = 36.9° or 0.64 rad. $R = \sqrt{(3^2 + 4^2)} = 5$. Hence:

Example

Express 6 sin ωt – 2.5 cos ωt in the form R sin ($\omega t + a$).

Using the double angle formula sin ($A + B$) = sin A cos B + cos A sin B:

$6 \sin \omega t - 2.5 \cos \omega t = R(\sin \omega t \cos a + \cos \omega t \sin a)$

Comparing coefficients of sin ωt gives $6 = R \cos a$ and of cos ωt gives $-2.5 = R \sin a$. Thus $\tan a = -2.5/6$. The negative sign for the sine and the tangent means that the angle must be in the fourth quadrant (see Figure 1.34). Hence a = –0.39 rad. $R = \sqrt{(6^2 + 2.5^2)} = 6.5$ and so:

$6 \sin \omega t - 2.5 \cos \omega t = 6.5 \sin (\omega t - 0.39)$

Thus, by subtracting the waveform 2.5 cos ωt from 6 sin ωt we end up with a waveform of amplitude 6.5 and a phase shift of –0.39 rad.

Example

Two sinusoidal alternating voltages of $v_1 = 1.25 \sin \omega t$ and $v_2 = 1.60 \cos \omega t$ are combined. Show that the result is a voltage of $v = 2 \sin (\omega t + 52°)$.

$v = v_1 + v_2 = 1.25 \sin \omega t + 1.60 \cos \omega t$

Using sin ($A + B$) = sin A cos B + cos A sin B, then:

$$2 \sin (\omega t + 52°) = 2(\sin \omega t \cos 52° + \cos \omega t \sin 52°)$$

$$= 1.23 \sin \omega t + 1.57 \cos \omega t$$

With the accuracy to which the result was quoted, the case is proved.

Adding phasors

Often in alternating current circuits we need to add the voltages across two components in series. We must take account of the possibility that the two voltages may not be in phase, despite having the same frequency since they are supplied by the same source. This means that if we consider the phasors, they will rotate with the same angular velocity but may have different lengths and start with a phase angle between them. Consider one of the voltages to have an amplitude V_1 and zero phase angle (Figure 1.50(a)) and the other an amplitude V_2 and a phase difference of ϕ from the first voltage (Figure 1.50 (b)). We can obtain the sum of the two by adding the two graphs, point-by-point, to obtain the result shown in Figure 1.50(c). Thus at the instant of time indicated in the figures, the two voltages are v_1 and v_2. Hence the total voltage is $v = v_1 + v_2$. We can repeat this for each instant of time and hence end up with the graph shown in Figure 1.50(c).

However, exactly the same result is obtained by adding the two phasors by means of the *parallelogram rule* of vectors. If we place the tails of the arrows representing the two phasors together and complete a parallelogram, then the diagonal of that parallelogram drawn from the junction of the two tails represents the sum of the two phasors. Figure 17.16(c) shows such a parallelogram and the resulting phasor with magnitude V.

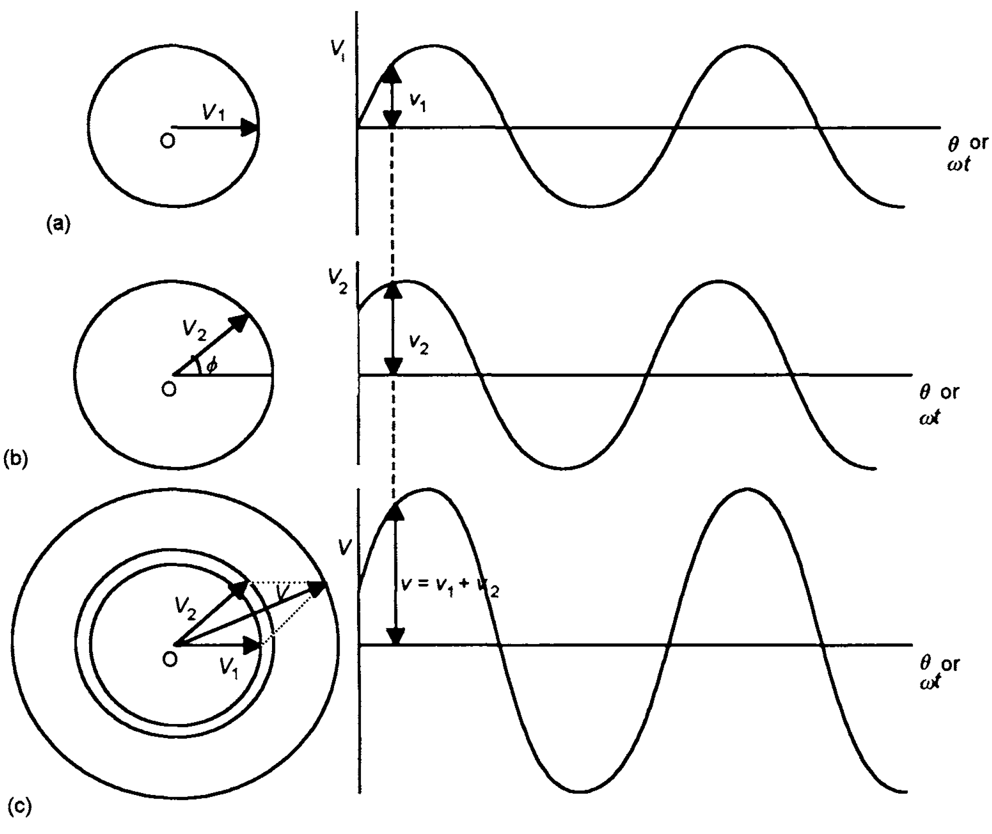


Figure 1.50 *Adding two sinusoidal signals of the same frequency*

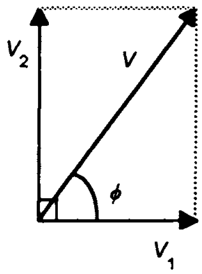


Figure 1.51 *Adding two phasors*

If the phase angle between the two phasors of sizes V_1 and V_2 is 90°, as in Figure 1.51, then the resultant can be calculated by the use of the Pythagoras theorem as having a size V of:

$$V^2 = V_1^2 + V_2^2 \qquad [35]$$

and is at a phase angle ϕ relative to the phasor for V_2 of:

$$\tan\phi = \frac{V_2}{V_1} \qquad [36]$$

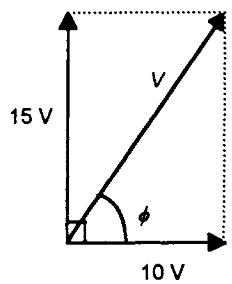


Figure 1.52 *Example*

Example

Two sinusoidal alternating voltages are described by the equations of v_1 = 10 sin ωt volts and v_2 = 15 sin (ωt + π/2) volts. Determine the sum of these voltages.

Figure 1.52 shows the phasor diagram for the two voltages. The angle between the phasors is π/2, i.e. 90°. We could determine the sum from a scale drawing or by calculation using the Pythagoras theorem. Thus:

$$(\text{sum})^2 = 10^2 + 15^2$$

Hence the magnitude of the sum of the two voltages is 18.0 V. The phase angle is given by:

$$\tan\phi = \frac{15}{10}$$

Hence ϕ = 56.3° or 0.983 rad. Thus the sum is an alternating voltage described by a phasor of amplitude 18.0 V and phase angle 56.3° (or 0.983 rad). This alternating voltage is thus described by:

v = 18.0 sin (ωt + 0.983) volts.

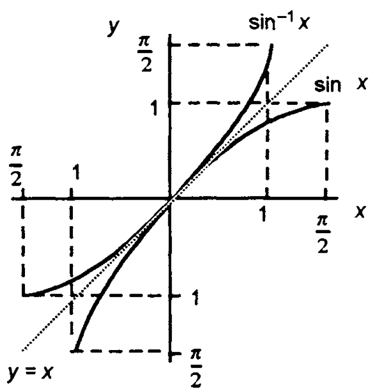


Figure 1.53 *sin x and its inverse*

1.5.4 The inverse circular functions

If sin x = 0.8 what is the value of x? This requires the inverse being obtained. There is an inverse if the function is one-to-one or restrictions imposed to give this state of affairs. However, the function y = sin x gives many values of x for the same value of y. To obtain an inverse we have to restrict the domain of the function to $-\pi/2$ to $+\pi/2$. With that restriction y = sin x has an inverse. The inverse function is denoted as $\sin^{-1} x$ (sometimes also written as arcsin x). Note that the -1 is *not* a power here but purely notation to indicate the inverse. If sin x = 0.8 then the value of x that gives this sine is the inverse and so $x = \sin^{-1} 0.8$, i.e. x = 53°. Figure 1.53 shows the graphs for sin x and its inverse function. In a similar way we can define inverses for cosines and tangents.

Key points

If y = sin x then $x = \sin^{-1} y$, when $-\pi/2 \le y \le \pi/2$.

If y = cos x then $x = \sin^{-1} y$, when $0 \le y \le \pi$.

If y = tan x then $x = \tan^{-1} y$, when $-\pi/2 \le y \le \pi/2$.

Problems 1.5

1 State the amplitude and phase angle (with respect to $y = 5 \sin \theta$) of the function $y = 5 \sin (\theta + 30°)$.

2 A cyclic function used to describe a rotating radius (phasor) is defined by the equation $y = 4 \sin 3t$. What is the amplitude and the angular frequency of the function?

3 State the amplitude, period and phase angle for the following cyclic functions:

(a) $2 \sin (5t + 1)$, (b) $6 \cos 3t$, (c) $5 \cos\left(\frac{2t+1}{3}\right)$,

(d) $2 \cos (t - 0.6)$

4 State the amplitude, period and phase angle for the following cyclic functions:

(a) $6 \sin (2t + 1)$, (b) $2 \cos 9t$, (c) $5 \cos\left(\frac{2t-1}{5}\right)$,

(d) $2 \cos (t - 0.2)$, (e) $5 \sin (4t + \pi/8)$, (f) $\frac{1}{2} \sin (t - \pi/0.6)$

5 The potential difference across a component in an electrical circuit is given by the equation $v = 40 \sin 40\pi t$. Deduce the maximum value of the potential difference and its frequency.

6 A sinusoidal voltage has a maximum value of 1 V and a frequency of 1 kHz. If the voltage has a phase angle of 60°, what will be the instantaneous voltage at times of (a) $t = 0$, (b) $t = 0.5$ ms?

7 A sinusoidal alternating current has an instantaneous value i at a time t, in seconds, given by $i = 100 \sin (200\pi t - 0.25)$ mA. Determine (a) the maximum current, (b) the frequency, (c) the phase angle.

8 A sinusoidal alternating voltage has an instantaneous value v at a time t, in seconds, given by $v = 12 \sin (100\pi t + 0.5)$ volts. Determine (a) the maximum voltage, (b) the frequency, (c) the phase angle.

9 What is the value of v, when $t = 30$ μs, for an amplitude-modulated radio wave with a voltage v in volts which varies with time t in seconds and is defined by the equation $v = 50(1 + 0.02 \sin 2400\pi t) \sin (200 \times 10^3 \pi t)$.

10 Show that $\sin (A + B + C) = \cos A \cos B \cos C (\tan A + \tan B + \tan C - \tan A \tan B \tan C)$.

11 Find the values of x between 0 and 360° which satisfy the condition $8 \cos x + 9 \sin x = 7.25$.

12 Write $5 \sin \theta + 4 \cos \theta$ in the forms (a) $R \sin (\theta - a)$, (b) $R \cos (\theta + a)$.

13 Express $W (\sin a + \mu \cos a)$ in the form $R \cos (a - \beta)$ giving the values of R and $\tan \beta$. Also show that the maximum value of the expression is $W\sqrt{(1 + \mu^2)}$ and that this occurs when $\tan a = 1/\mu$.

14 Write the following functions in the form $R \sin (\omega t + a)$: (a) $3 \sin \omega t + 4 \cos \omega t$, (b) $4.6 \sin \omega t - 7.3 \cos \omega t$,

(c) $-2.7 \sin \omega t - 4.1 \cos \omega t$

15 Express $3 \sin \theta + 5 \cos \theta$ in the form $R \sin (\theta + a)$ with a measured in degrees.

16 Write the following functions in the form $R \sin (\omega t \pm a)$:
(a) $4 \sin \omega t - 3 \cos \omega t$, (b) $-7 \sin \omega t + 4 \cos \omega t$,
(c) $-3 \sin \omega t - 6 \cos \omega t$

17 The currents in two parallel branches of a circuit are $10 \sin \omega t$ milliamps and $20 \sin (\omega t + \pi/2)$ milliamps. What is the total current entering the parallel arrangement?

18 The voltage across a component in a circuit is $5.0 \sin \omega t$ volts and across another component in series with it $2.0 \sin (\omega t + \pi/6)$ volts. Determine the total voltage across both components.

19 The sinusoidal alternating voltage across a component in a circuit is $50\sqrt{2} \sin (\omega t + 40°)$ volts and across another component in series with it $100\sqrt{2} \sin (\omega t - 30°)$ volts. What is the total voltage across the two components?

20 The currents in two parallel branches of a circuit are $4\sqrt{2} \sin \omega t$ amps and $6\sqrt{2} \sin (\omega t - \pi/3)$ amps. What is the total current entering the parallel arrangement?

21 Determine the value in radians of:
(a) $\sin^{-1} 0.74$, (b) $\cos^{-1} 0.10$, (c) $\tan^{-1} 0.80$, (d) $\sin^{-1} 0.40$

1.6 Exponential functions

There are many situations in engineering where we are concerned with functions which grow or decay with time, e.g.

- The variation with time of the temperature of a cooling object.
- The variation with time of the charge on a capacitor when it is being charged and when it is being discharged.
- The variation with time of the current in a circuit containing inductance when the current is first switched on and then when it is switched off.
- The decay with time of the radioactivity of a radioactive isotope.

This section is about the equations we can use to describe such growth or decay.

Exponentials

In general, we can describe growth and decay processes by an equation of the form:

$$y = a^t \qquad [37]$$

where a is some constant called the *base*, and y the value of the quantity at a time t. Thus, for growth, we might have 2^t, 3^t, 4^t, etc. and for decay 2^{-t}, 3^{-t}, 4^{-t}, etc. We could write equations for growth or decay processes with different values of the base. However, we

usually standardise the base to one particular value. The most widely used form of equation is e^x, where e is a constant with the value 2.728 281 828 ... Whenever an engineer refers to an exponential change he or she is almost invariably referring to an equation written in terms of e^x. Why choose this strange number 2.718... for the base? The reason is linked to the properties of expressions written in this way. For $y = e^x$, the rate of change of y with x, i.e. the slope of a graph of y against x, is equal to e^x (this is discussed in more detail in the chapter concerned with differentiation):

$$\text{slope of graph of } y \text{ against } x = y = e^x \qquad [38]$$

and there are many engineering situations where this property occurs.

A simple illustration of the above is given if we take a strip of paper and cut it into half, throwing away one of the halves. We then take the half strip and cut it into half, throwing away one of the halves. If we keep on repeating this procedure we obtain the graph shown in Figure 1.54(a). This is an exponential decay in the length of the paper. Now look at the change in length per tear, i.e. the 'gradient' of the graph, Figure 1.54(b). We have the same exponential function. A similar type of relationship exists in the discharge of a charged capacitor. The charge on the capacitor decreases exponentially with time and the rate of change of charge, i.e. the current, follows the same exponential decay.

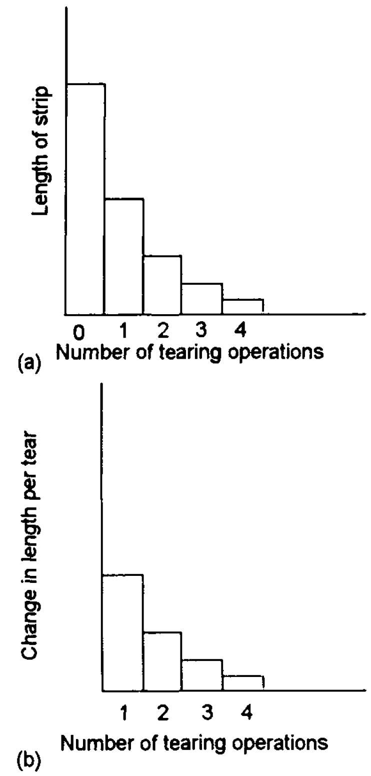


Figure 1.54 *An 'exponential decay'*

The following shows the values of e^x and e^{-x} for various values of x and Figure 1.55 the resulting graphs

$y = e^x$	0.14	0.37	1	2.72	7.39	20.09	... infinite
x	−2	−1	0	1	2	3	... infinite

$y = e^{-x}$	7.39	2.72	1	0.37	0.14	0.05	... 0
x	−2	−1	0	1	2	3	... infinite

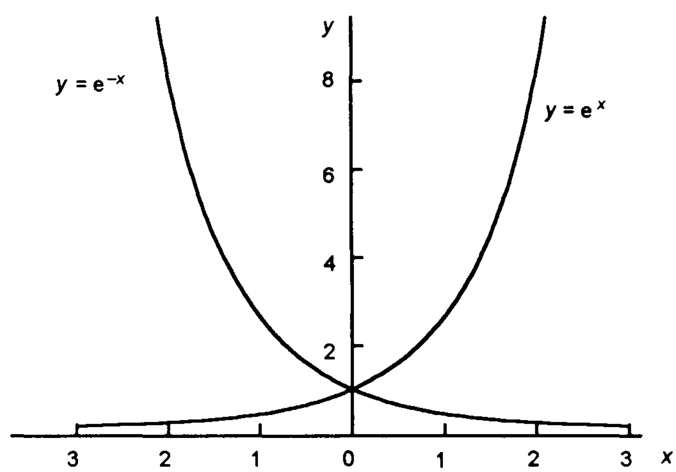


Figure 1.55 $y = e^x$ *and* $y = e^{-x}$

The e^x graph describes a growth curve, the e^{-x} a decay curve. Note that both graphs have $y = 1$ when $x = 0$.

In a more general form we can write the exponential equation in the form $y = e^{kx}$, or $y = e^{-kx}$, where k is some constant. This constant k determines how fast y changes with x. The following data illustrates this:

x	0	1	2	3	... infinite
$y = e^{-1x}$	1	0.368	0.135	0.050	... 0
$y = e^{-2x}$	1	0.135	0.018	0.003	... 0

x	0	1	2	3	... infinite
$y = e^{1x}$	1	2.718	7.389	20.086	... infinite
$y = e^{2x}$	1	7.389	54.598	403.429	... infinite

The bigger k is the faster y decreases, or increases, with x.

When $x = 0$ then for $y = e^{kx}$, or $y = e^{-kx}$, $y = e^0$ and so $y = 1$. This is thus the value of y that occurs when x is zero. Since we may often have an initial value other than 1, we write the equation in the form:

$$y = A\, e^{kx} \qquad [39]$$

where A is the initial value of y at $x = 0$. For example, for the discharging of a capacitor in an electrical circuit we have, for the charge q on the capacitor at a time t, the equation:

$$q = Q_0\, e^{-t/CR} \qquad [40]$$

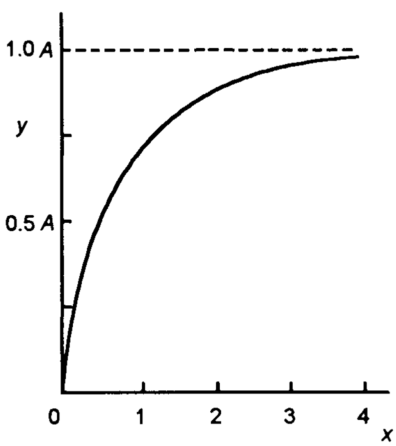


Figure 1.56 $y = A - A\, e^{-kx}$

When $t = 0$ then $q = Q_0$. The constant k is $1/CR$. The bigger the value of CR the smaller the value of $1/CR$ and so the slower the rate at which the capacitor becomes discharged.

One form of equation involving exponentials that is quite common is of the form:

$$y = A - A\, e^{-kx} \qquad [41]$$

When $x = 0$ then $e^0 = 1$ and so $y = A - A = 0$. The initial value is thus 0. As x increases then e^{-kx} decreases from 1 towards 0, eventually becoming zero when x is infinite. Thus the value of y increases as x increases. When x is very large then e^{-kx} becomes virtually 0 and so y becomes equal to A. Figure 8.4 shows the graph. It shows a quantity y which increases rapidly at first and then slows down to become eventually A.

For example, for a capacitor which starts with zero charge on its plates and is then charged we have the equation:

$$q = Q_0 - Q_0\, e^{-t/CR} \qquad [42]$$

When $t = 0$ then $e^0 = 1$ and so $q = Q_0 - Q_0 = 0$. As t increases, so the value of $e^{-t/CR}$ decreases and so q becomes more and more equal to Q_0.

Example

For an object cooling according to Newton's law, the temperature θ of the object varies with time t according to the equation $\theta = \theta_0 e^{-kt}$, where θ_0 and k are constants. (a) Explain why this equation represents a quantity which is decreasing with time. (b) What is the value of the temperature at $t = 0$? (c) How will the rates at which the object cools change if in one instance $k = 0.01$ and in another $k = 0.02$ (the units of k are per °C)?

(a) If we assume that t and k will only have positive values, then the $-kt$ means that the power is negative and so the temperature decreases with time.
(b) When $t = 0$ then $e^{-kt} = 1$ and so $\theta = \theta_0$. Thus θ_0 is the initial value at the time $t = 0$.
(c) Doubling the value of k means that the object will cool faster, in fact it will cool twice as fast.

Example

The current i in amperes in an electrical circuit varies with time t according to the equation $i = 10(1 - e^{-t/0.4})$. What will be (a) the initial value of the current when $t = 0$, (b) the final value of the current at infinite time?

(a) When $t = 0$ then $e^{-t/0.4} = e^0 = 1$. Thus $i = 10(1 - 1) = 0$.
(b) When t becomes very large then $e^{-t/0.4}$ becomes 0. Thus we have $i = 10(1 - 0)$ and so the current becomes 10 A.

Maths in action

Time constant

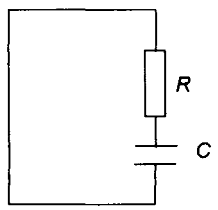


Figure 1.57 *Discharge of a charged capacitor*

Consider the discharging of a charged capacitor through a resistance (Figure 1.57). The voltage v_C across the capacitor varies with time t according to the equation $v_C = V e^{-RC}$, where V is the initial potential difference across the capacitor at time $t = 0$. Suppose we let $\tau = RC$, calling τ the *time constant* for the circuit. Thus, $v_C = V e^{-t/\tau}$. The time taken for v_C to drop from V to $0.5V$ is thus given by:

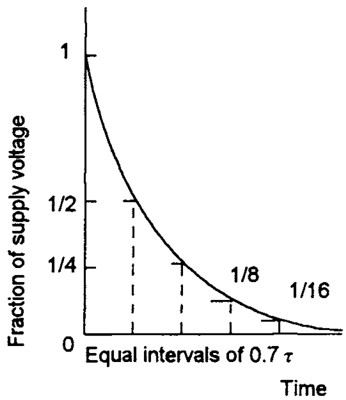


Figure 1.58 *Voltage across the capacitor*

Discharge of a capacitor

Time	v_C
0	V
$0.7T$	$0.5V$
$1.4T$	$0.25V$
$2.1T$	$0.125V$
$2.8T$	$0.0625V$
$3.5T$	$0.03125V$

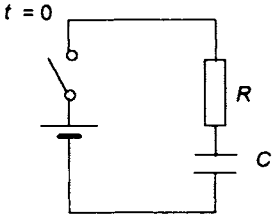


Figure 1.59 *Charging a capacitor*

$0.5V = V e^{-t/\tau}$

$e^{-t/\tau} = 0.5$

$-\frac{t}{\tau} = \ln 0.5 = -0.693$

Thus in a time of 0.693τ the voltage will drop to half its initial voltage. The time taken to drop to $0.25V$ is given by:

$0.25V = V e^{-t/\tau}$

$e^{-t/\tau} = 0.25$

$-\frac{t}{\tau} = \ln 0.25 = -1.386$

Thus in a time of 1.386τ the voltage will drop to one-quarter of its initial voltage. This is twice the time taken to drop to half the voltage. This is a characteristic of a decaying exponential graph: if t is the time taken to reach half the steady-state value, then in $2t$ it will reach one-quarter, in $3t$ it will reach one-eighth, etc. In each of these time intervals it reduces its value by a half (Figure 1.58).

When $t = 1\tau$ then $v_C = V e^{-1} = 0.632V$. Thus in a time equal to the time constant the voltage across the capacitor drops to 63.2% of the initial voltage. When $t = 2\tau$ then $v_C = V e^{-2} = 0.135V$. Thus the voltage across the capacitor drops to 13.5% of the initial voltage. When $t = 3\tau$ then $v_C = V e^{-3} = 0.050V$. Thus the voltage across the capacitor drops to 5.0% of the initial voltage.

Now consider the growth of the charge on an initially uncharged capacitor when a voltage is switched across it (Figure 1.59). The time constant τ is RC. Thus:

$v_C = V(1 - e^{-t/RC}) = V(1 - e^{-t/\tau})$

What time will be required for v_C to reach $0.5V$?

$0.5V = V(1 - e^{-t/\tau})$

$e^{-t/\tau} = 0.5$

$-\frac{t}{\tau} = \ln 0.5 = -0.693$

Thus in a time of 0.693τ the voltage will reach half its steady-state voltage. The time taken to reach $0.75V$ is given by:

$0.75V = V(1 - e^{-t/\tau})$

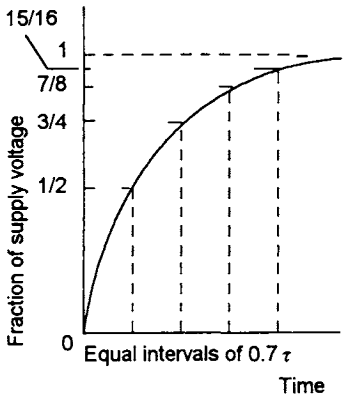


Figure 1.60 *Voltage across the capacitor*

Growth of the p.d. across C

Time	v_C
0	0
$0.7T$	$0.5V$
$1.4T$	$0.75V$
$2.1T$	$0.875V$
$2.8T$	$0.938V$
$3.5T$	$0.969V$

$e^{-t/\tau} = 0.25$

$-\frac{t}{\tau} = \ln 0.25 = -1.386$

Thus in a time of 1.386τ the voltage will reach three-quarters of its steady-state value. This is twice the time taken to reach half the steady-state voltage. This is a characteristic of exponential graphs: if t is the time taken to reach half the steady-state value, then in $2t$ it will reach three-quarters, in $3t$ it will reach seven-eighths, etc. In each successive time interval of 0.7τ the p.d. across the capacitor reduces its value by a half (Figure 1.60).

When $t = 1\tau$ then $v_C = V(1 - e^{-1}) = 0.632V$. Thus in a time equal to the time constant the voltage across the capacitor rises to 63.2% of the steady-state voltage. When $t = 2\tau$ then $v_C = V(1 - e^{-2}) = 0.865V$. Thus the voltage across the capacitor rises to 86.5% of the steady-state voltage. When $t = 3\tau$ then $v_C = V(1 - e^{-3}) = 0.950V$. Thus the voltage across the capacitor rises to 95.0% of the steady-state voltage.

Damped oscillations

In Section 1.5 we considered the vertical oscillations of a mass on the end of a spring (Figure 1.22) with Figure 1.24 showing how the vertical displacement of the mass can be described by a sinusoidal oscillation with an amplitude which decays with time. In the absence of damping the displacement is described by:

$y = A \sin \omega t$

where the amplitude A is a constant. With the damped oscillation we replace the constant A by a term involving exponential decay, i.e.

$$y = C\,e^{-\zeta\omega t} \sin \omega t \qquad [43]$$

with C being a constant and ζ a damping term called the *damping factor*. At zero time the exponential term has the value 1 and so C is the initial amplitude. As the time increases so the exponential term becomes smaller and smaller and the amplitude term thus decreases.

Key points

$a^x a^y = a^{x+y}$

$\frac{a^x}{a^y} = a^{x-y}$

$(a^x)^y = a^{xy}$

$(ab)^x = a^x b^x$

$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

where a and b are bases.

1.6.1 Manipulating exponentials

The techniques used for the manipulation of exponentials are the same as those for manipulating powers. The following examples illustrate this.

Example

Simplify the following:

(a) $e^{2t}e^{4t}$, (b) $(e^{2t})^{-3}$, (c) $\frac{e^{5x}}{e^{2x}}$, (d) $\frac{10\,e^{t/2}}{2\,e^{t/3}}$, (e) $\frac{1}{e^{2t}}+\frac{2}{e^{3t}}$

(a) $e^{2t}e^{4t} = e^{2t+4t} = e^{6t}$

(b) $(e^{2t})^{-3} = e^{-6t}$

(c) $\frac{e^{5x}}{e^{2x}} = e^{5x-2x} = e^{3x}$

(d) $\frac{10\,e^{t/2}}{2\,e^{t/3}} = 5\,e^{\frac{t}{2}-\frac{t}{3}} = 5\,e^{t/6}$

(e) Bringing the fraction to a common denominator:

$$\frac{1}{e^{2t}}+\frac{2}{e^{3t}} = \frac{e^{3t}+2\,e^{3t}}{e^{2t}e^{3t}} = \frac{e^{3t}+2\,e^{3t}}{e^{5t}} = e^{-2t}+2\,e^{-3t}$$

Alternatively we could take the reciprocals of each term and write the equation as:

$$\frac{1}{e^{2t}}+\frac{2}{e^{3t}} = e^{-2t}+2\,e^{-3t}$$

Problems 1.6

1 The number N of radioactive atoms in a sample of radioactive material decreases with time t and is described by the equation $N = N_0\,e^{-\lambda t}$, where N_0 and λ are constants. (a) Explain why this equation represents a quantity which is decreasing with time. (b) What will be the number of radioactive atoms at time $t = 0$? (c) For a radioactive material that decreases only very slowly with time, will λ have a large or smaller value than with a radioactive material which decreases quickly with time?

2 The length L of a rod of material increases from some initial length with the temperature θ above that at which the initial length is measured and is described by the equation $L = L_0\,e^{a\theta}$, where L_0 and a are constants. (a) Explain why the equation represents a quantity which increases with time. (b) What will be the length of the rod when h = 0? (c) What will be the effect of a material having a higher value of a than some other material?

3 For an electrical circuit involving inductance, the current in amperes is related to the time t by the equation $i = 3(1 - e^{-10t})$. What is the value of the current when (a) $t = 0$, and (b) t is very large?

4 What are the values of y in the following equations when (i) $x = 0$, (ii) x is very large, i.e. infinite?

(a) $y = 2\ e^{3x}$, (b) $y = 10\ e^{-5x}$, (c) $y = 2(1 - e^{-2x})$,

(d) $y = 2\ e^{-0.2x}$, (e) $y = -4\ e^{-x/3}$, (f) $y = 0.5(1 - e^{-x/5})$,

(g) $y = 4(1 - e^{-x/2})$, (h) $y = 10\ e^{4x}$, (i) $y = 0.2 - 0.2\ e^{-3x}$

5 The voltage, in volts, across a capacitor is given by $20\ e^{-0.1t}$, where t is the time in seconds. Determine the voltage when t is (a) 1 s, (b) 10 s.

6 The atmospheric pressure p is related to the height h above the ground at which it is measured by the equation $p = p_0\ e^{-h/c}$, where c is a constant and p_0 the pressure at ground level where $h = 0$. Determine the pressure at a height of 1000 m if p_0 is 1.01×10^5 Pa and $c = 70\ 000$ (unit m^{-1}).

7 The current i, in amperes, in a circuit involving an inductor in series with a resistor when a voltage is E is applied to the circuit at time $t = 0$ is given by the equation

$$i = \frac{E}{R}(1 - e^{-Rt/L})$$

If R/L has the value 2 Ω/H (actually the same unit as seconds), what is the current when (a) $t = 0$, (b) $t = 1$ s?

8 The voltage v across a resistor in series with an inductor when a voltage E is applied to the circuit at time $t = 0$ is given by the equation $v = E(1 - e^{-t/T})$, where T is the so-called time constant of the circuit. If $T = 0.5$ s, what is the voltage when (a) $t = 0$, (b) $t = 1$ s?

9 The charge q on a discharging capacitor is related to the time t by the equation $q = Q_0\ e^{-t/CR}$, where Q_0 is the charge at $t = 0$, R is the resistance in the circuit and C the capacitance. Determine the charge on a capacitor after a time of 0.2 s if initially the charge was 1 μC (1 μC $= 10^{-6}$ C), R is 1 MΩ and C is 4 μF. Note that with the units in seconds (s), coulombs (C), ohms (Ω) and farads (F), the resulting charge will be in coulombs.

10 The current i, in amperes, in a circuit with an inductor in series with a resistor is given by the equation $i = 4(1 - e^{-10t})$, where the time t is in seconds. Determine the current when (a) $t = 0$, (b) $t = 0.05$ s, (c) $t = 0.10$ s, (d) $t = 0.15$ s, (e) t = infinity.

11 The voltage v, in volts, across a capacitor after a time t, in seconds, is given by the equation $v = 10\ e^{-t/3}$. Determine the value of the voltage v after 2 s.

12 The resistance R, in ohms, of an electrical conductor at a temperature of θ°C is given by the equation $R = R_0\ e^{a\theta}$. Determine the resistance at a temperature of 1000°C if R_0 is 5000 Ω and a is 1.2×10^{-4} (unit per °C).

13 The current i, in amperes, in an electrical circuit varies with time t and is given by the equation $i = 2(1 - e^{-10t})$. Determine the current after times of (a) 0.1 s, (b) 0.2 s, (c) 0.3 s.

14 The amount N of a radioactive material decays with time t and is given by the equation $N = N_0 \, e^{-0.7t}$, where t is in years. If at time $t = 0$ the amount of radioactive material is 1 g, what will be the amount after five years?

15 The atmospheric pressure p, in pascals, varies with the height h, in kilometres, above sea level according to the equation $p = p_0 \, e^{-0.15h}$. If the pressure at sea level is 10^5 Pa, what will be the pressure at heights of (a) 1 km, (b) 2 km?

16 The voltage v, in volts, across an inductor in an electrical circuit varies with time t, in milliseconds, according to the equation $v = 200 \, e^{-t/10}$. Determine the voltage after times of (a) 0.1 ms, (b) 0.5 ms.

17 When the voltage E to a circuit consisting of an inductor in series with a resistor is switched off, the voltage across the inductor varies with time t according to the equation $v = -E \, e^{-t/T}$, where T is the time constant of the circuit. If $T = 2$ s, determine the voltage when (a) $t = 0$, (b) $t = 1$ s.

18 When a voltage E is applied to a circuit consisting of a capacitor in series with a resistor at time $t = 0$, the voltage v across the capacitor varies with time according to the equation $v = E(1 - e^{-t/T})$, where T is the time constant of the circuit. If $T = 0.1$ s, determine the voltage when (a) $t = 0$, (b) $t = 0.1$ s.

19 The temperature θ, in °C, of a cooling object varies with time t, in minutes, according to $\theta = 200 \, e^{-0.04t}$. Determine the temperature when (a) $t = 0$, (b) $t = 10$ minutes, (c) t is infinite.

20 Under one set of conditions the amplitude A of the oscillations of a system varies with time t according to the equation $A = A_0 \, e^{kt}$. Under other conditions the amplitude varies according to the equation $y = A_0 \, e^{-kt}$. If k is a positive number, how do the oscillations differ?

21 Simplify the following:

(a) e^3e^5, (b) $e^{3t}e^{5t}$, (c) $e^{-5t}e^{3t}$, (d) $(e^{-4t})^3$, (e) $(1 + e^{2t})^2$,

(f) $\frac{1}{e^{3t}}$, (g) $\frac{e^{3t}}{e^{5t}}$, (h) $\left(\frac{1}{e^{4t}}\right)^2$, (i) $\frac{10\,e^{4t}}{2\,e^{t}}$.

1.7 Log functions

Consider the function $y = 2^x$. If we are given a value of x then we can determine the corresponding value of y. However, suppose we are given a value of y and asked to find the value of x that could have produced it. The inverse function is called the *logarithm function* and is defined, for $y = a^x$ and $a > 0$, as:

$$x = \log_a y \qquad [44]$$

This is stated as 'log to base a of y equals x'. Thus, if we take an input of x to a function $f(x) = a^x$ and then follow it by the inverse

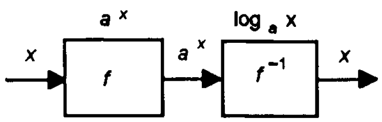


Figure 1.61 $f(x)f^{-1}(x) = x$

Key points

The defining equations for logs is:

$\log_a a^x = x$

Most logarithms use base 10 or base e. Logarithms to base 10 are often just written as log or lg, the base 10 being then understood. Logarithms to base e are termed *natural logarithms* and often just written as ln.

function $f^{-1}(x) = \log_a(x)$, as in Figure 1.61, then because it is an inverse we obtain x and so:

$$\log_a a^x = x \qquad [45]$$

While logarithms can be to any base, most logarithms use base 10 or base e. Logarithms to base 10 are often just written as log or lg, the base 10 being then understood. Logarithms to base e are termed *natural logarithms* and often just written as ln. Figure 1.61 shows the graph of $y = e^x$ and its inverse of the natural logarithm function.

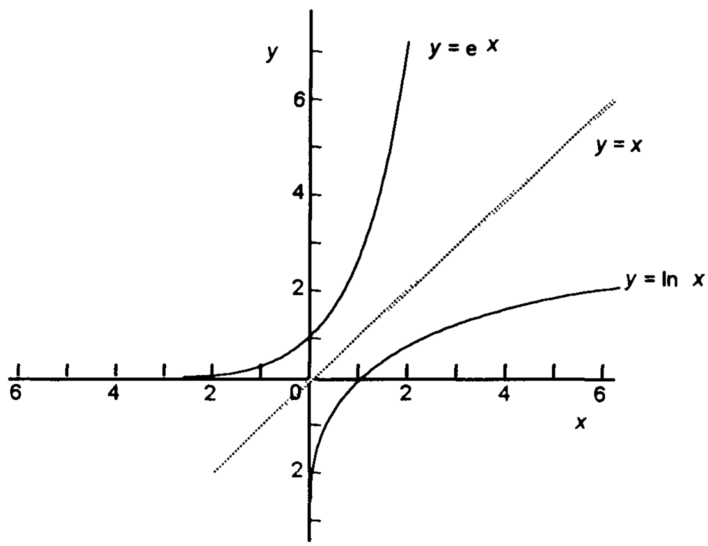


Figure 1.62 *The exponential and its inverse of the natural logarithm function*

Since $a^{A+B} = a^A a^B$ then:

$$\log_a A + \log_a B = \log_a AB \qquad [46]$$

$$n \log_a A = \log_a(A^n) \qquad [47]$$

Since $a^{A-B} = \frac{a^A}{a^B}$ then:

$$\log_a A - \log_a B = \log_a \frac{A}{B} \qquad [48]$$

Since $a^1 = a$ then $\log_a a = 1$.

Sometimes there is a need to change from one base to another, e.g. $\log_a x$ to $\log_b x$. Let $u = \log_b x$ then $b^u = x$ and so taking logarithms to base a of both sides gives $\log_a b^u = \log_a x$ and so $u \log_a b = \log_a x$. Since $u = \log_b x$ then $(\log_b x)(\log_a b) = \log_a x$ and so:

$$\log_b x = \frac{\log_a x}{\log_a b} \qquad [49]$$

Example

Write $\lg\left(\frac{\sqrt{a}}{bc^3}\right)$ in terms of lg a, lg b and lg c.

We have:

$$\lg\left(\frac{\sqrt{a}}{bc^3}\right) = \lg\sqrt{a} - \lg(bc^3)$$

Hence:

$$\lg\left(\frac{\sqrt{a}}{bc^3}\right) = \tfrac{1}{2}\lg a - \lg b - 3\lg c$$

Example

Simplify (a) lg x + lg x^3, (b) 3 ln x + ln(1/x).

(a) $\lg x + \lg x^3 = \lg(x \times x^3) = \lg x^4$
(b) $3 \ln x + \ln(1/x) = \ln x^3 + \ln(1/x) = \ln(x^3/x) = \ln x^2$

Example

Solve for x the equation $2^{2x-1} = 12$.

Taking logarithms of both sides of the equation gives:

$(2x - 1) \lg 2 = \lg 12$

Hence:

$$2x - 1 = \frac{\lg 12}{\lg 2} = 3.58$$

Thus $x = 2.29$.

The decibel

The power gain of a system is the ratio of the output power to the input power. If we have, say, three systems in series (Figure 1.63) then the power gain of each system is given by:

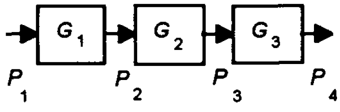


Figure 1.63 *Systems in series*

$$G_1 = \frac{P_2}{P_1},\ G_2 = \frac{P_3}{P_2},\ G_3 = \frac{P_4}{P_3}$$

The overall power gain of the system is P_4/P_1 and is the product of the individual gains, i.e.

$$G = \frac{P_4}{P_1} = \frac{P_2}{P_1} \times \frac{P_3}{P_2} \times \frac{P_4}{P_3} = G_1 \times G_2 \times G_3 \qquad [50]$$

Taking logarithms gives:

$$\lg G = \lg G_1 + \lg G_2 + \lg G_3 \qquad [51]$$

We thus can add the log ratio of the powers. This log of the power ratio was said to be the power ratio in units of the *bel*, named in honour of Alexander Graham Bell:

$$\text{Power ratio in bels} = \lg \frac{\text{power out}}{\text{power in}} \qquad [52]$$

Thus the overall power gain in bels can be determined by simply adding together the power gains in bels of each of the series systems. The bel is an inconveniently large quantity and thus the *decibel* is used:

$$\text{Power ratio in decibels} = 10 \lg \frac{\text{power out}}{\text{power in}} \qquad [53]$$

A power gain of 3 dB is thus a power ratio of 2.0.

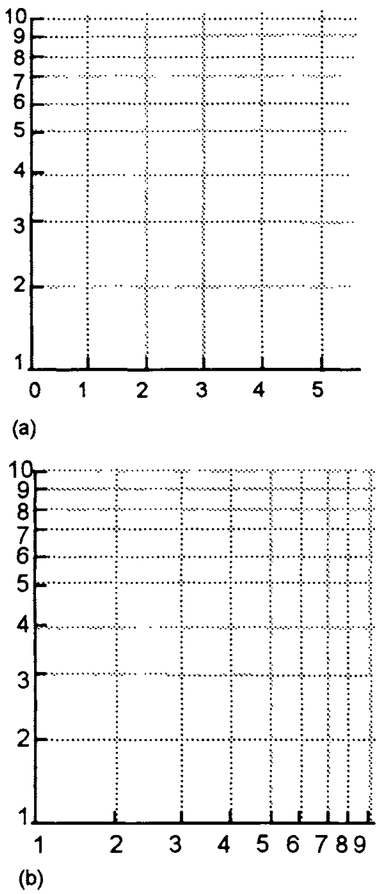


Figure 1.64 *(a) Log-linear and (b) log-log graph paper*

Log graphs

When a graph is a straight line then the relationship between the two variables can be stated as being of the form $y = mx + c$ and we can easily determine the constants m and c from the graph and hence obtain the relationship. However, if we have a relationship of the form $y = ax^b$, where a and b are constants, then a plot of y against x gives a non-linear graph from which it is not easy to determine a and b. However, we can write the equation as:

$$\lg y = \lg x^b + \lg a = b \lg x + \lg a \qquad [54]$$

A graph of $\lg y$ against $\lg x$ will thus be a straight line graph with a gradient of b and an intercept of $\lg a$. Likewise, if we have the relationship $y = a\,e^{bx}$ then, taking logarithms to base e:

$$\ln y = \ln e^{bx} + \ln a = bx + \ln a \qquad [55]$$

A graph of $\ln y$ against x will give a straight line graph with a gradient of b and an intercept of $\ln a$.

To avoid having to take the logarithms of quantities, it is possible to use special graph paper which effectively takes the logarithms for you. Figure 1.64 shows the form taken by log-linear and log-log graph paper. On a logarithmic scale, the distance between 1 and 10 is the same as between 10 and 100, each of these distances being termed a cycle.

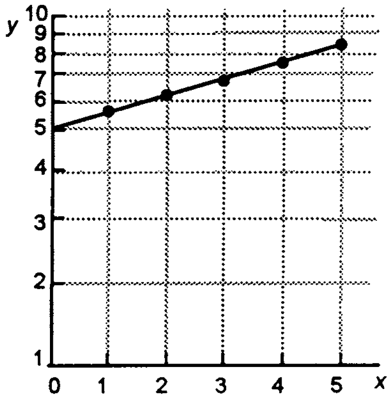


Figure 1.65 *Example*

Example

It is believed that the relationship between y and x for the following data is of the form $y = a\ e^{bx}$. Show that this is the case and determine, using log-linear graph paper, the values of a and b.

y	5.53	6.11	6.75	7.46	8.24
x	1	2	3	4	5

Taking logarithms to base e gives $\ln y = bx + \ln a$. We thus require log-linear graph paper. The y-axis, which is the ln axis, has to range from $\ln 5.53 = 1.7$ to $\ln 8.24 = 2.1$ and so just one cycle from 1 to 10 is required. Figure 1.65 shows the resulting graph. The graph is straight line and so the relationship is valid. The gradient is

$$\text{gradient} = \frac{\ln 8.44 - \ln 5.53}{5-1} = \frac{2.13-1.71}{4} = 0.10$$

The intercept with the y-axis, i.e. $x = 0$ ordinate, is at 5. Thus the required equation is $y = 5\ e^{0.10x}$.

Key point

Before actually plotting graphs, or creating spreadsheets to plot graphs, it is useful to first sketch the 'form' that the graph might be expected to have in order to get the 'feel' of what the actual plotted graph should look like.

Example

The relationship between power P (in watts), the e.m.f. E (in volts) and the resistance R (in ohms) is thought to be of the form $P = E^n/R$. In a test in which R was kept constant, the following measurements were recorded:

E (volts)	5	10	15	20	25	30
P (watts)	2.5	10	22.5	40	62.5	90

Determine whether the above relationship is true (or approximately so) and determine the values for n and R.

Taking ln of both sides of the equation gives:

$$\ln P = \ln\left(\frac{E^n}{R}\right) = \ln E^n - \ln R = n \ln E - \ln R$$

So, if the relationship is true, a graph of $\ln P$ against $\ln E$ should be a straight line. The values of $\ln P$ and $\ln E$ are:

$\ln E$	1.61	2.30	2.71	3.00	3.22	3.40
$\ln P$	0.92	2.30	3.11	3.69	4.14	4.50

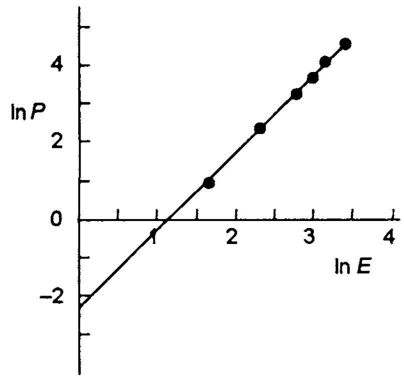


Figure 1.66 *Example*

Figure 1.66 shows the plot. From the graph we obtain an intercept on the y-axis of −2.3 and a gradient of about 2.

We thus have $-2.3 = -\ln R$ and so:

$\ln R = 2.3$

$R = e^{2.3}$

and $R = 9.9$, or 10 when rounded up. With $n = 2$ we thus have:

$$P = \frac{E^2}{10}$$

We can test that this is valid by choosing any two results from the test, e.g. $E = 5$ V, $P = 2.5$ W and substituting them into the equation. With $E = 5$ V the equation gives $P = 23/10 = 2.5$ V and so the test confirms the equation.

Maths in action

Radioactive materials, e.g. uranium 235, decay and the mass of that isotope decreases with time. The rate of decay of the isotope is proportional to the mass of isotope present:

rate of decay = $-\lambda m$

where λ is a constant called the decay constant. If m_0 is the mass at time $t = 0$ and mass m the mass at time t, then the following relationship can be derived from the above equation:

$m = m_0\, e^{-\lambda t}$

Taking ln gives:

$\ln m = -\lambda t + \ln m_0$

A graph of $\ln m$ plotted against t will be a straight line graph of slope $-\lambda$ and intercept $+\ln m_0$.

Problems 1.7

1 Simplify (a) $2 \lg x + \log x^2$, (b) $\ln 2x^3 - \ln(4/x^2)$.

2 Write the following in terms of $\lg a$, $\lg b$ and $\lg c$:

(a) $\lg\left(\frac{b\sqrt{2}}{ac}\right)$, (b) $\lg\left(\frac{ab}{\sqrt{c}}\right)^3$

3 Solve for x the equations: (a) $3^x = 300$, (b) $10^{2-3x} = 6000$, (c) $7^{2x+1} = 4^{3-x}$.

4 The following data indicates how the voltage v across a component in an electrical circuit varies with time t. It is considered that the relationship between V and t might be of the form $v = V\,e^{-bt}$. Show that this is so and determine the values of V and b.

v in volts	3.75	1.38	0.51	0.19	0.07
t in s	10	20	30	40	50

5 A hot object cools with time. The following data shows how the temperature θ of the object varies with time t. The relationship between θ and t is expected to be of the form $\theta = a\,e^{-bx}$. Show that this is so and determine the values of a and b.

θ in °C	536	359	241	162	108
t in min	2	4	6	8	10

6 The rate of flow Q of water over a V-shaped notch weir was measured for different heights h of the water above the point of the V and the following data obtained. The relationship between Q and h is thought to be of the form $Q = ah^b$. Show that this is so and determine the values of a and b.

Q in m³/s	0.13	0.26	0.46	2.12	1.07
h in m	0.3	0.4	0.5	0.6	0.7

7 The amplitude A of oscillation of a pendulum decreases with time t and gives the following data. Show that the relationship is of the form $A = a\,e^{bt}$ and determine the values of a and b.

A in mm	268	180	120	81	54
t in s	20	40	60	80	100

8 The tension T and T_0 in the two sides of a belt driving a pulley and in contact with the pulley over an angle of q is given by the equation $T = T_0\,e^{\mu\theta}$. Determine the values of T_0 and μ for the following data:

T in N	69.5	80.8	91.1	109.1	126.7
θ in radians	1.1	1.6	2.0	2.6	3.1

9 In an electrical circuit, the current i in mA occurring when an 8.3 μF capacitor is being discharged varies with time t in ms as shown in the following table:

i (mA)	50.0	17.0	5.8	1.7	0.58	0.24
t (ms)	200	255	310	375	425	475

If I and T are constants, with I being the initial current in mA, show that the above results are connected by the equation $i = I\,e^{t/T}$ and determine I and T.

10 The pressure P at a height h above ground level is given by $P = P_0\,e^{-h/c}$, where P_0 is the pressure at ground level and c is a constant. When P_0 is 1.013×10^5 Pa and the pressure at a height of 1570 m is 9.871×10^4 Pa, determine graphically the value of c.

1.8 Hyperbolic functions

Key points

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

When we want to describe the curve a rope hangs in we use, what is termed, an hyperbolic function. The sine, cosine and tangent are termed circular functions because their definition is associated with a circle. In a similar way, the sinh (pronounced sinch or shine), cosh (pronounced cosh) and tanh (pronounced than or tanch) are *hyperbolic functions* associated with a hyperbola. Sinh is a contracted form of 'hyperbolic sine', cosh of 'hyperbolic cosine' and tanh of 'hyperbolic tangent'. Figure 1.67 shows the comparison of the circular and hyperbolic functions. The hyperbolic functions are defined as:

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) \qquad [56]$$

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}) \qquad [57]$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad [58]$$

Also we have sech x = 1/cosh x, cosech x = 1/sinh x and coth x = 1/tanh x.

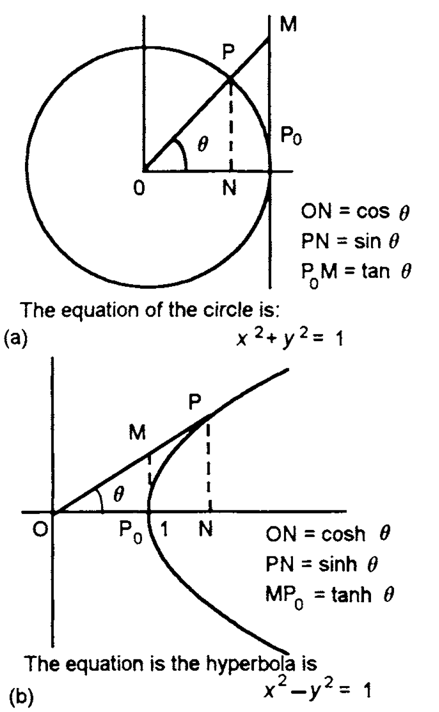


Figure 1.67 *(a) Circular functions, (b) hyperbolic functions*

Example

Determine, using a calculator, the values of (a) cosh 3, (b) sinh 3.

Some calculators have hyperbolic functions so that they can be evaluated by the simple pressing of a key, with others you will have to evaluate the exponentials.

(a) Evaluating the exponentials:

$\cosh 3 = \frac{1}{2}(e^3 + e^{-3}) = 10.07.$

(b) Evaluating the exponentials:

$\sinh 3 = \frac{1}{2}(e^3 - e^{-3}) = 10.02.$

1.8.1 Graphs of hyperbolic functions

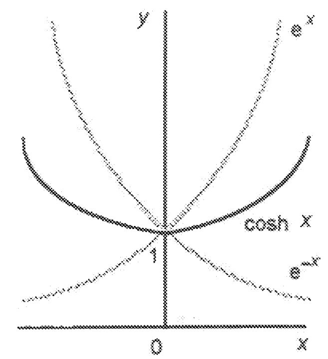


Figure 1.68 *cosh x*

Since cosh x is the average value of e^x and e^{-x} we can obtain a graph of cosh x as a function of x by plotting the e^x and e^{-x} graphs and taking the average value. Figure 1.68 illustrates this. Note that unlike cos x, cosh x is not a periodic function. At $x = 0$, cosh $x = 1$. The curve is symmetrical about the y-axis, i.e. cosh $(-x)$ = cosh x and is termed an even function.

To obtain the graph of sinh x from those of e^x and e^{-x}, at a particular value of x we subtract the second from the first and then take half the resulting value. Figure 1.69 illustrates this. Note that unlike sin x, sinh x is not a periodic function. When $x = 0$, sinh x = 0. The curve is symmetrical about the origin, i.e. sinh($-x$) = $-$sinh x, and is said to be an odd function.

Figure 1.70 shows the graph of tanh x, obtained by taking values of e^x and e^{-x} and calculating values of tanh x for particular values of x. Unlike tan x, tanh x is not periodic. When $x = 0$, tanh $x = 0$. All the values of tanh x lie between -1 and $+1$. As x tends to infinity, tanh x tends to 1. As x tends to minus infinity, tanh x tends to -1. The curve is symmetrical about the origin, i.e. tanh($-x$) = $-$tanh x, and is said to be an odd function.

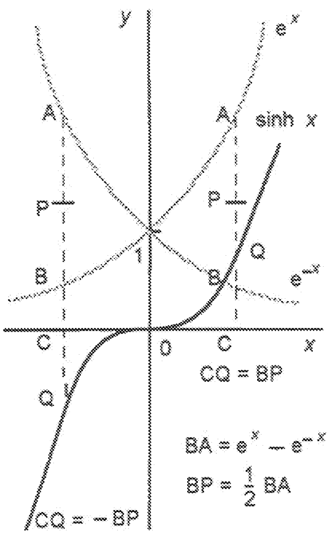


Figure 1.69 *sinh x*

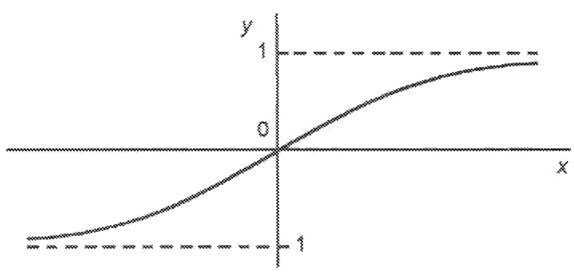


Figure 1.70 $y = \tanh x$

Maths in action

Hyperbolic functions and suspended cables

Often it is necessary for engineers to analyse frameworks in order to test for their integrity, i.e. safety and ability to function as designed under a range of conditions. The design engineer needs to formulate a mathematical 'model' which will accurately represent the real system when built. Such problems often involve hyperbolic functions as the following example shows.

Consider a uniform cable which is suspended from two fixing points A and B and which hangs under its own weight (Figure 1.71). Point A is higher than point B and the cable has a uniform weight μ per unit length.

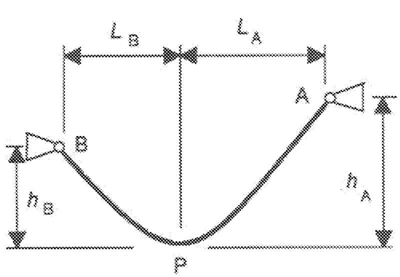


Figure 1.71 *Sagging cable*

By drawing free-body diagrams for the forces involved on an element of the cable and considering its equilibrium we can arrive at a differential equation (see Chapter 4 for a discussion of such equations), which when solved leads to the equation for the gradient a distance x from P:

$$\text{gradient} = \frac{e^{\mu x/T_0} - e^{-\mu x/T_0}}{2} = \sinh\left[\frac{\mu x}{T_0}\right]$$

T_0 is the horizontal component of the tension in the cable at P. This equation can then give (by integration, see Chapter 4) the height y above P of the cable at distance x as:

$$y = \frac{T_0}{\mu}\cosh\left(\frac{\mu x}{T_0}\right) + k$$

where k is a constant. Since $x = 0$ when $y = 0$, we can put these values in the equation and obtain $k = -T_0/\mu$. Thus:

$$y = \frac{T_0}{\mu}\left[\cosh\left(\frac{\mu x}{T_0}\right) - 1\right]$$

This is the equation of the curve of the cable, known as a catenary. For a full analysis of the system, see the companion book in this series: *Mechanical Engineering Systems* by R. Gentle, P. Edwards and W. Bolton

Problems 1.8

1 Determine, using a calculator, the values of (a) sinh 2, (b) cosh 5, (c) tanh 2, (d) sinh(–2), (e) cosech 1.4, (f) sech 0.8.

2 A flexible cable suspended between two horizontal points hangs in the form of a catenary (Figure 1.72), the equation of the curve being given by $y = c[\cosh(x/c) - 1]$, where y is the sag of the cable, x the horizontal distance from the midpoint to one end of the cable and c is a constant. Determine the sag of a cable when $c = 20$ and $2x = 16$ m.

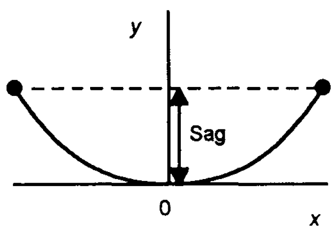


Figure 1.72 *Problem 2*

3 The speed v of a surface wave on a liquid is given by:

$$v = \sqrt{\left[\left(\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\rho\lambda}\right)\tanh\frac{2\pi h}{\lambda}\right]}$$

where g is the acceleration due to gravity, λ the wavelength of the waves, γ the surface tension, ρ the density and h the depth of the water. What will the speed approximately be for (a) shallow water waves when h/k tends to zero, (b) deep water waves when h/k tends to infinity?

2 Vectors, phasors and complex numbers

Summary

Vectors are means by which engineers describe quantities which need both a direction and a magnitude specified if their effects are to be ascertained. Vectors play a strong part in the formulation and analysis of mechanical systems, both static and dynamic. Phasors are a means by which sinusoidal alternating voltages and currents can be specified in terms of a rotating radius and an angle, they behaving like vectors. This chapter looks at how we can work with such quantities, considering both vector algebra and complex numbers.

Objectives

By the end of this chapter, the reader should be able to:

- add and subtract vector quantities;
- use vector components to add and subtract vectors;
- use phasors to describe sinusoidal alternating voltages and currents;
- represent phasors by polar notation and be able to work with quantities expressed in this way;
- represent phasors by complex numbers and work with quantities expressed in this way.

2.1 Vectors

If we talk of the mass of this book then we quote just a number, this being all that is needed to give a specification of its mass. However, if we quote a force then in order to fully describe the force we need to specify both its size and the direction in which it acts. Quantities which are fully specified by a statement of purely size are termed *scalars*. Quantities for which we need to specify both size and direction in order to give a full specification are termed *vectors*. Examples of scalar quantities are mass, distance, speed, work and energy. Examples of vector quantities are displacement, velocity, acceleration and force.

> **Key points**
>
> A scalar quantity is defined by purely its magnitude; a vector quantity has to have both its magnitude and direction defined.

To specify a vector we need to specify its magnitude and direction. Thus, we can represent it by a line segment AB (Figure 2.1) with a length which represents the magnitude of the vector and a direction, indicated by the arrow on the segment, which

Figure 2.1 *Representing a vector*

represents the direction of the vector. We can denote this vector representation as

$$\overrightarrow{AB}$$

the arrow indicating the direction of the line segment being from A to B. Note that:

$$\overrightarrow{AB} \neq \overrightarrow{BA}$$

One of the vectors is directed from A to B while the other is directed from B to A. An alternative notation is often used, lower case bold notation **a** being used in print, or underlining $\underline{a}$ in writing. With this notation, if we write **a** or $\underline{a}$ from the vector from A to B then the vector from B to A is represented as $-\mathbf{a}$ or $-\underline{a}$, the minus sign being used to indicate the vector is in the opposite direction.

The length of the line segment represents the *magnitude* of the vector. This is indicated by the notation:

$$|\overline{AB}| \quad \text{or} \quad |\mathbf{a}| \quad \text{or} \quad a$$

Unit vector

A vector which is defined as having a magnitude of 1 is termed a *unit vector*, such a vector often being denoted by the symbol $\hat{\mathbf{a}}$.

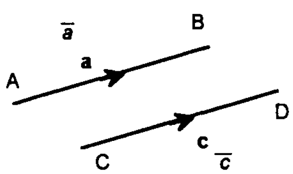


Figure 2.2 *Equal vectors*

Like vectors

Two vectors are equal if they have the same magnitude and direction. Thus the vectors in Figure 2.2 are equal, even if their locations differ. A vector is only defined in terms of its magnitude and direction, its location is not used in its specification. Thus, for Figure 2.2, we can write:

$$\overrightarrow{AB} = \overrightarrow{CD} \quad \text{or} \quad \mathbf{a} = \mathbf{c} \quad \text{or} \quad \underline{a} = \underline{c}$$

Multiplication of vectors by a number

If a vector is multiplied by a positive real number k then the result is another vector with the same direction but with a magnitude that is k times the original magnitude. This is multiplication of a vector by a scalar.

$$k \times \mathbf{a} = k\mathbf{a} \qquad [1]$$

We can consider a vector **a** with magnitude $|\mathbf{a}|$ as being a unit vector, i.e. a vector with a magnitude 1, multiplied by the magnitude $|\mathbf{a}|$ (note that the magnitude $|\mathbf{a}|$ is a scalar), i.e.

$$\mathbf{a} = |\mathbf{a}|\hat{\mathbf{a}} \qquad [2]$$

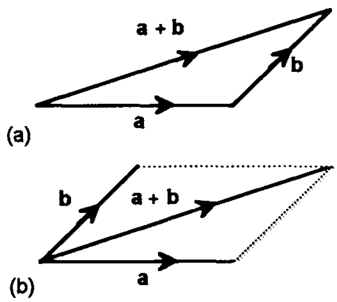


Figure 2.3 *(a) Triangle rule, (b) parallelogram rules*

Key points

The *triangle rule* can be stated as: to add two vectors **a** and **b** we place the tail of the line segment representing one vector at the head of the line segment representing the other and the line that forms the third side of the triangle represents the vector sum of **a** and **b**.

The *parallelogram* rule can be stated as: to add two vectors **a** and **b** we place the tails of the line segments representing the vectors together and then draw lines parallel to them to complete a parallelogram, the diagonal of the parallelogram drawn from the initial junction of the two tails represents the vector sum of a and b

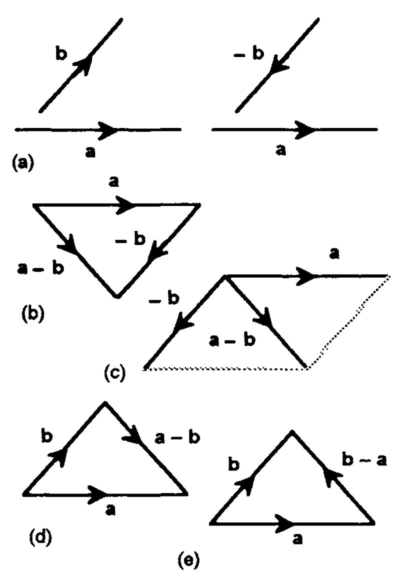


Figure 2.4 *(a) The vectors, (b) subtraction by the triangle rule, (c) subtraction by the parallelogram rule (d)* **a** – **b**, *(e)* **b** – **a**

Maths in action

In the vector analysis of a mechanical system, we can write Newton's 2nd Law of Motion as a vector equation:

F = *ma*

where **F** is the resultant force acting on a system and **a** is the resulting acceleration. The equation is a vector equation since the direction of the acceleration must be in the same direction as the force; both the force and the acceleration are vector quantities. Newton's 1st Law contains the principle of equilibrium of forces and is used in the following section concerned with the addition and subtraction of vectors; we have to bother about both magnitude and direction to consider equilibrium. Newton's 3rd Law is basic to our understanding of force, stating that forces always occur in pairs with equal in magnitude but opposite in direction forces.

2.1.1 Adding and subtracting vectors

Consider the following situation involving displacement vectors. An aeroplane flies 100 km due west, then 60 km in a north-westerly direction. What is the resultant displacement of the aeroplane from its start point? If the initial displacement vector is **a** and the second displacement vector is **b**, then what is required is the vector sum **a** + **b**.

One way we can determine the sum of two vectors involves the *triangle rule* and is shown in Figure 2.3(a). Note that **a** and **b** have directions that go in one sense round the triangle and the sum **a** + **b** has a direction in the opposite sense. An alternative way of determining the sum involves the *parallelogram rule* and is shown in Figure 2.3(b).

Subtraction of vector **b** from **a** is carried out by adding –**b** to **a**:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) \qquad [3]$$

The addition of **a** and –**b** is carried out using the triangle (Figure 2.4(b)) or parallelogram rules (Figure 2.4(c)). Note that, whatever rule we use, the vector **a** – **b** can be represented by the vector from the end point of **b** to the end point of **a** (Figure 2.4(d)), the vector from the end point of **a** to the end point of **b** being **b** – **a** (Figure 2.4(e)).

The triangle rule for the addition of vectors can be extended to the addition of any number of vectors. If the vectors are represented in magnitude and direction by the sides of a *polygon* then their sum is represented in magnitude and direction by the

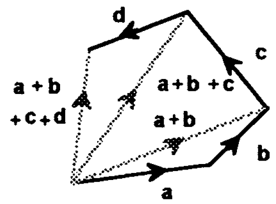


Figure 2.5 *Polygon of vectors*

line segment used to close the polygon (Figure 2.5). Essentially what we are doing is determining the sum of vector 1 and vector 2 using the triangle, then adding to this sum vector 3 by a further triangle and repeating this for all the vectors.

If we have a number of vectors and the vectors give a closed triangle or polygon, then, since the line segment needed to close the figure has zero length, the sum of the vectors must be a vector with no magnitude. This is a statement of equilibrium.

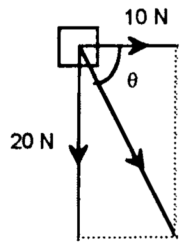


Figure 2.6 *Example*

Example

An object is acted on by two forces, one of which is has a size of 10 N and acts horizontally and the other a size of 20 N which acts vertically. Determine the resultant force.

Figure 2.6 shows the vectors and the use of the parallelogram rule to determine the sum. We can calculate, using the Pythagoras theorem, the diagonal as a having a size of $\sqrt{(20^2 + 10^2)} = 22.4$ N. It is at an angle θ to the horizontal force, with $\theta = \tan^{-1}(20/10) = 63.4°$.

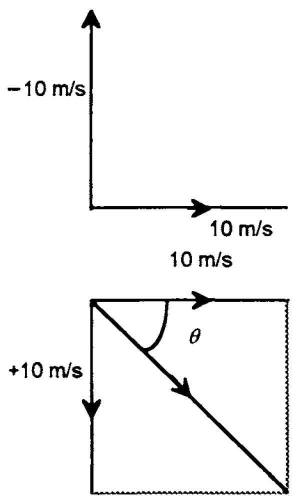


Figure 2.7 *Example*

Example

Determine the resultant velocity if we have velocities of 10 m/s acting horizontally to the right and −10 m/s acting vertically upwards.

This problem requires the addition of two vectors, Figure 2.7 showing the vectors and the use of the parallelogram rule to determine the sum. A −10 m/s vector upwards is the same as a +10 m/s vector downwards. Hence, the magnitude of the sum, i.e. the diagonal of the parallelogram, is given by the Pythagoras theorem as $\sqrt{(10^2 + 10^2)} = 14.1$ m/s and it is at an angle below the horizontal of θ where $\theta = \tan^{-1}(10/10) = 45°$.

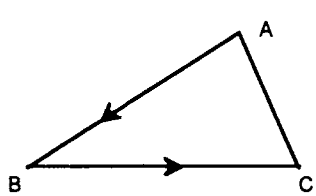


Figure 2.8 *Example*

Example

For the triangle ABC (Figure 2.8) if **a** is the vector from A to B and **b** the vector from B to C, express the vector from C to A in terms of **a** and **b**.

Using the triangle rule: $\overrightarrow{AC} = \overrightarrow{BC} + \overrightarrow{AB}$.

Since $\overrightarrow{CA} = -\overrightarrow{AC}$, then we have: $\overrightarrow{CA} = -(\mathbf{a} + \mathbf{b})$.

2.1.2 Components

In mechanics a common technique to aid in the solution of problems is to replace a single vector by two components which are at right angles to each other, generally in the horizontal and the vertical directions. Then we can sum all the horizontal components, sum all the vertical components, and are then left with the simple problem of determining the resultant of two vectors at right angles to each other.

For the vector **a** in Figure 2.9 we have **h** and **v** as the horizontal and vertical components. Thus for the magnitudes we must have:

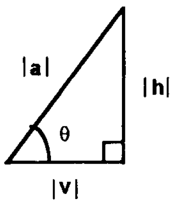


Figure 2.9 *Resolution of a vector into two components*

$$|\mathbf{h}| = |\mathbf{a}| \cos \theta \qquad [4]$$

$$|\mathbf{v}| = |\mathbf{a}| \sin \theta \qquad [5]$$

Example

Express a force of 10 N at 40° to the horizontal in terms of horizontal and vertical components.

Horizontal component = 10 cos 40° = 7.7 N
Vertical component = 10 sin 40° = 6.4 N

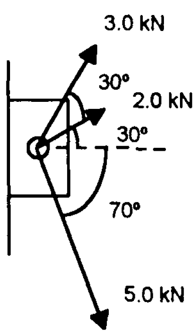


Figure 2.10 *Example*

Example

Determine the resultant force acting on the bracket shown in Figure 2.10 due to the three forces indicated.

For the 3 kN force we have:

horizontal component = 3.0 cos 60° = 1.5 kN
vertical component = 3.0 sin 60° = 2.6 kN

For the 2.0 kN force we have:

horizontal component = 2.0 cos 30° = 1.7 kN
vertical component = 2.0 sin 30° = 1.0 kN

For the 5.0 kN force we have:

horizontal component = 5.0 cos 70° = 1.7 kN
vertical component = –5.0 sin 70° = –4.7 kN

The minus sign is because this force is acting downwards and in the opposite direction to the other vertical components which we have taken as being positive. All the horizontal components are in the same direction. Thus:

sum of horizontal components = 1.5 + 1.7 + 1.7 = 4.9 kN
sum of vertical components = 2.6 + 1.0 – 4.7 = –1.1 kN

Figure 2.11 shows how we can use the parallelogram rule to find the resultant with these two components. Since the two components are at right angles to each other, the resultant can be calculated using the Pythagoras theorem. Thus, the magnitude of the resultant is:

resultant = $\sqrt{(4.9^2 + 1.1^2)}$ = 5.0 kN

The resultant is at an angle θ downwards from the horizontal given by:

$$\tan\theta = \frac{1.1}{4.9}$$

Thus θ = 12.7°.

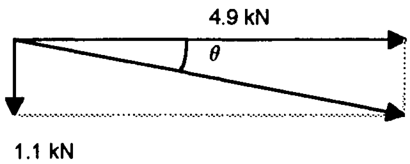


Figure 2.11 *Example*

Components in terms of unit vectors

A useful way of tackling problems involving summing vectors by considering their components is to write them in terms of unit vectors. Consider the x-y plane shown in Figure 2.12. Point P has the coordinates (x, y) and is joined to the origin O by the line OP. This line from O to P can be considered to be a vector **r** anchored at O and specifying a position, being defined by its two components **a** and **b** along the x and y directions with:

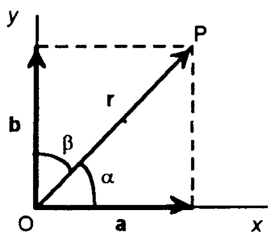


Figure 2.12 *Components*

$$\mathbf{r} = \mathbf{a} + \mathbf{b}$$

If we define **i** to be a unit vector along the x-axis then $\mathbf{a} = a\mathbf{i}$, where a is the magnitude of the **a** vector. If we define **j** to be a unit vector along the y-axis then $\mathbf{b} = b\mathbf{j}$, where b is the magnitude of the **b** vector. Thus:

$$\mathbf{r} = a\mathbf{i} + b\mathbf{j}$$

But a is the x-coordinate of P and b the y-coordinate of P. Thus we can write:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \qquad [6]$$

For example, we might specify a position vector as $3\mathbf{i} + 2\mathbf{j}$. This would mean a position vector from the origin to a point with the coordinates (3, 2).

The magnitude of the vector **r** is given by the Pythagoras theorem as:

$$|\mathbf{r}| = \sqrt{x^2 + y^2} \qquad [7]$$

Key points

The term *position vector* is used for a vector that emanates from or is directed towards a particular point. Vectors for which the location is not significant are termed *free vectors*.

A unit vector may be formed by dividing a vector by its magnitude.

If a and β are the angles the vector **r** makes with the x- and y-axes, then:

$$\cos a = \frac{x}{|\mathbf{r}|} \quad \text{and} \quad \cos\beta = \frac{y}{|\mathbf{r}|} \qquad [8]$$

These are known as the *direction cosines* of **r**.

Example

If **r** = 4**i** + 7**j** determine |**r**| and the angle **r** makes with the x-axis.

$$|\mathbf{r}| = \sqrt{4^2 + 7^2} = 8.1$$

The angle with the x-axis is given by:

$$\cos a = \frac{4}{8.1}$$

Thus the angle is 60.4°.

Example

Figure 2.13 shows three forces F_1, F_2 and F_3 all acting at a single point A on a wall bracket. In order to calculate the pulling force on the bracket at the wall, so that it can be safely connected to the wall when under load, determine the size of the force components of F_1, F_2 and F_3 in the x and y directions.

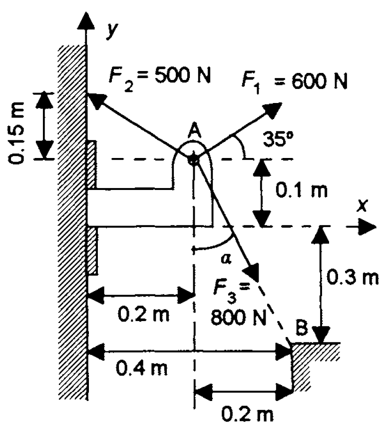


Figure 2.13 *Example*

The components of F_1 in the x and y directions are:

F_{1x} = 600 cos 35° = 491 N

F_{1y} = 600 sin 35° = 344 N

The components of F_2 in the x and y directions are (the vector forms the hypotenuse of a 3-4-5 triangle):

F_{2x} = –500 (4/5) = 400 N

F_{2y} = 500 (3/5) = 300 N

The components of F_3 in the x and y directions are, with a = tan^{-1} (0.2/0.4) = 26.6°:

F_{3x} = 800 sin 26.6° = 358 N

F_{3y} = –800 cos 26.6° = 715 N

Alternatively, the size of the components of F_3 may be obtained by writing F_3 at a magnitude times a unit vector $\mathbf{r}_{AB}$ in the direction of A to B. The position vector is 0.2**i** – 0.4**j** and its magnitude is $\sqrt{(0.2^2 + 0.4^2)} = 0.447$. Thus the unit vector is (0.2**i** – 0.4**j**)/0.447 and so:

$F_3 = 800(0.447\mathbf{i} - 0.896\mathbf{j}) = 358\mathbf{i} - 715\mathbf{j}$ N

and so $F_{3x} = 358$ N and $F_{3y} = -715$ N.

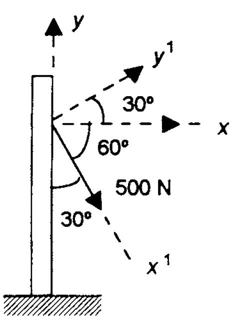


Figure 2.14 *Example*

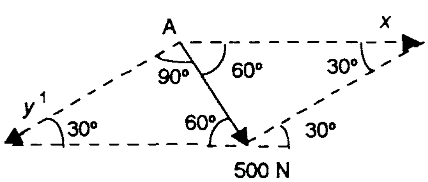


Figure 2.15 *Example*

Key point

The *sine rule*: For a triangle, the length of a side *a* divided by the sine of the opposite angle *A* equals the length of side *b* divided by the sine of its opposite angle *B*.

Example

In a structural test, a 500 N force was applied to a vertical pole, as shown in Figure 2.14. (a) Write the 500 N force in terms of the unit vectors **i** and **j** and identify its *x* and *y* components. (b) Determine the components of the 500 N force along the x^1 and y^1 directions. (c) Determine the components of the 500 N force along the *x* and y^1 directions.

(a) $\mathbf{F} = (500 \cos 60°)\mathbf{i} - (500 \sin 60°)\mathbf{j} = 250\mathbf{i} - 433\mathbf{j}$

Thus the vector components are $\mathbf{F}x = 250\mathbf{i}$ N and $\mathbf{F}y = -433\mathbf{j}$ N.

(b) Axis y^1 is at 90° to x^1 and so, since the 500 N is in the x^1 direction we have the component in the $x1$ direction as 500 N and in the y^1 direction as 0.

(c) Here the required directions are not at right angles to each other and so we determine them by using the parallelogram rule. Figure 2.15 shows the parallelogram. If we use the sine rule:

$$\frac{|\mathbf{F}_x|}{\sin 90°} = \frac{500}{\sin 30°}$$

Hence the size of the *x* component is 1000 N.

$$\frac{|\mathbf{F}_{y^1}|}{\sin 60°} = \frac{500}{\sin 30°}$$

Hence the size of the y^1 component is 866 N. The two components are thus 1000 N and –866 N.

Addition and subtraction of vectors

Consider the addition of the two position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ shown in Figure 2.16, P having the coordinates (x_1, y_1) and Q the coordinates (x_2, y_2). Thus:

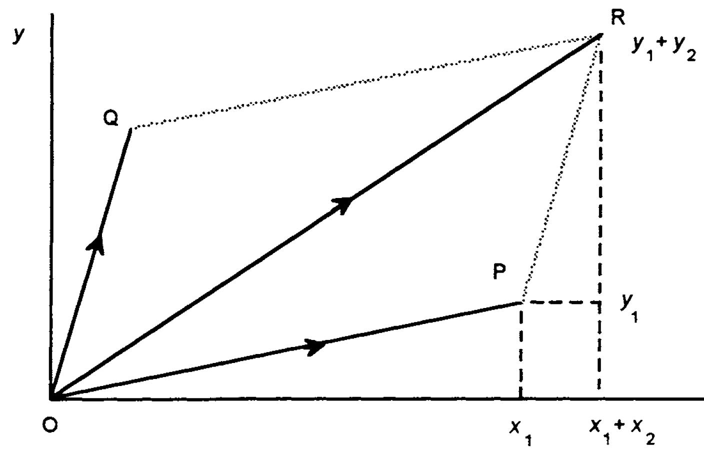


Figure 2.16 *Adding position vectors*

$\overrightarrow{OP} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\overrightarrow{OQ} = x_2\mathbf{i} + y_2\mathbf{j}$

We can obtain the sum by the use of the parallelogram rule as $\overrightarrow{OR}$. R has the coordinates $(x_1 + x_2, y_1 + y_2)$. Thus:

$$\overrightarrow{OR} = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} \qquad [9]$$

Key point

Adding or subtracting position vectors is achieved by adding or subtracting their respective co-ordinates.

Example

If **a** = 2i + 4j and **b** = 3i + 5j, determine (a) **a** + **b**, (b) **a** – **b**, (c) **a** + 2**b**.

(a) **a** + **b** = (2 + 3)**i** + (4 + 5) **j** = 5**i** + 8**j**
(b) **a** – **b** = (2 – 3)**i** + (4 – 5) **j** = –1**i** + –1**j**
(c) **a** + 2**b** = (2 + 6)**i** + (4 + 10) **j** = 8**i** + 14**j**

Example

ABCD is a square. If forces of magnitudes 1 N, 2 N and 3 N act parallel to AB, BC and CD respectively, in the directions indicated by the order of the letters, determine the magnitude and direction of the resultant force.

Figure 2.17 shows the directions of the forces. Expressing the forces in terms of unit vector components then the force parallel to AB is 1**i**, parallel to BC is 2**j** and that parallel to CD is –3**i**. Thus the resultant is 1**i** + 2**j** – 3**i** = –2**i** + 2**j** N. This will have a magnitude $\sqrt{[(-2)^2 + 2^2]}$ = 2.8 N at an angle of $\tan^{-1}(2/-2)$ = 135° to AB.

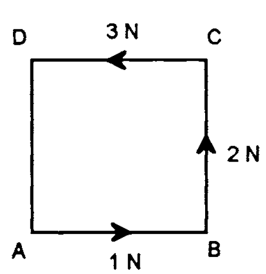


Figure 2.17 *Example*

Example

Forces of 5**i** – 5**j** N and –1**i** + 3**j** N act on a particle of mass 2 kg. Determine the resulting acceleration.

The resultant force is 5**i** – 5**j** – 1**i** + 3**j** = 4**i** – 2**j** N. Thus:

F = 4**i** – 2**j** = m**a** = 2**a**

Hence **a** = 2**i** – 1**j** m/s^2 and so the acceleration has a magnitude of $\sqrt{[2^2 + (-1)^2]}$ = 2.2 m/s^2 and is at an angle of $\tan^{-1}(-1/2) = -26.6°$ to the **i** direction.

Vectors in space

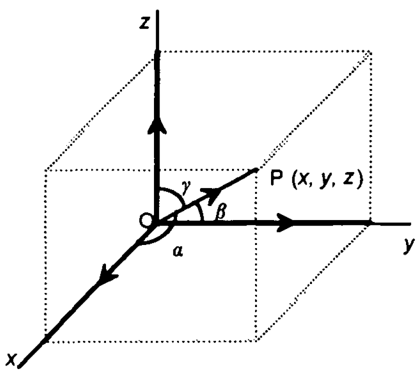


Figure 2.18 *Vector in space*

Here we extend the consideration of components to three dimensions (Figure 2.18). A vector **r** from O to P, with coordinates (x, y, z), is then defined by its vector components in the three mutually perpendicular directions x, y and z. If **i**, **j** and **k** are the unit vectors in the directions x, y and z, then:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad [10]$$

The magnitude of **r** is given by:

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \qquad [11]$$

The direction of a vector in three dimensions is determined by the angles it makes with the three axes, x, y and z, i.e. the angles a, β and γ. With (x, y, z) the coordinates of the position vector:

$$\cos a = \frac{x}{|\mathbf{r}|}, \quad \cos\beta = \frac{y}{|\mathbf{r}|} \text{ and } \cos\gamma = \frac{z}{|\mathbf{r}|} \qquad [12]$$

These are termed the *direction cosines*. As with the two-dimensional case, the basic rule for position vectors is: *adding or subtracting position vectors is achieved by adding or subtracting their respective coordinates.*

Example

Determine the magnitude and the direction cosines of the vector **r** = 2**i** + 3**j** + 6**k**.

Magnitude = $|\mathbf{r}| = \sqrt{2^2 + 3^2 + 6^2} = 7$

The direction cosines are:

$$l = \cos a = \frac{2}{7}, \quad m = \cos\beta = \frac{3}{7}, \quad n = \cos\gamma = \frac{6}{7}$$

> **Example**
>
> If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}$, determine (a) $\mathbf{a} + \mathbf{b}$, (b) $\mathbf{a} - \mathbf{b}$, (c) $\mathbf{a} + 2\mathbf{b}$.
>
> (a) $\mathbf{a} + \mathbf{b} = (2 + 3)\mathbf{i} + (3 - 2)\mathbf{j} + (4 + 1)\mathbf{k} = 5\mathbf{i} + 1\mathbf{j} + 5\mathbf{k}$
>
> (b) $\mathbf{a} - \mathbf{b} = (2 - 3)\mathbf{i} + (3 + 2)\mathbf{j} + (4 - 1)\mathbf{k} = -1\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$
>
> (c) $\mathbf{a} + 2\mathbf{b} = (2 + 6)\mathbf{i} + (3 - 4)\mathbf{j} + (4 + 2)\mathbf{k} = 8\mathbf{i} - 1\mathbf{j} + 6\mathbf{k}$

Problems 2.1

1. If vector **a** is a velocity of 3 m/s in a north-westerly direction and **b** a velocity of 5 m/s in a westerly direction, determine: (a) $\mathbf{a} + \mathbf{b}$, (b) $\mathbf{a} - \mathbf{b}$, (c) $\mathbf{a} - 2\mathbf{b}$.
2. If vector **a** is a displacement of 5 m in a northerly direction and **b** a displacement of 12 m in an easterly direction, determine: (a) $\mathbf{a} + \mathbf{b}$, (b) $\mathbf{a} - \mathbf{b}$, (c) $\mathbf{b} - \mathbf{a}$, (d) $\mathbf{a} + 2\mathbf{b}$.
3. ABCD is a quadrilateral. Determine the single vector which is equivalent to:

   (a) $\vec{AB} + \vec{BC}$, (b) $\vec{BC} + \vec{CD}$, (c) $\vec{AB} + \vec{DA}$.

4. If O, A, B, C and D are five points on a plane and $\vec{OA}$ represents the vector **a**, $\vec{OB}$ the vector **b**, $\vec{OC}$ the vector $\mathbf{a} + 2\mathbf{b}$, and $\vec{OD}$ the vector $2\mathbf{a} - \mathbf{b}$, express (a) $\vec{AB}$, (b) $\vec{BC}$, (c) $\vec{CD}$, and (d) $\vec{AC}$ in terms of **a** and **b**.
5. ABCD is a square. A force of 6 N acts along AB, 5 N along BC, 7 N along DB and 9 N along CA. Determine the resultant force.
6. Determine the vector sums of:

   (a) $\vec{AB} + \vec{BC} + \vec{CD}$, (b) $\vec{AB} - \vec{CB} + \vec{CD} + \vec{DE}$,

   (c) $\vec{AB} + \vec{BC} - \vec{DC} - \vec{AD}$, (d) $\vec{AB} + \vec{BC} + \vec{CD} + \vec{DC}$

7. A point is acted on by two forces, a force of 6 N acting horizontally and a force of 4 N at 20° to the horizontal. Determine the resultant components of the forces in the vertical and horizontal directions.
8. For the following vectors determine their magnitudes and angles to the *x*-axis: (a) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j}$, (b) $\mathbf{r} = 5\mathbf{i} + 2\mathbf{j}$, (c) $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j}$.
9. If $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j}$, determine: (a) $\mathbf{a} + \mathbf{b}$, (b) $\mathbf{a} - \mathbf{b}$, (c) $\mathbf{a} + 2\mathbf{b}$.
10. If $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$, determine: (a) $\mathbf{a} + \mathbf{b}$, (b) $\mathbf{a} - \mathbf{b}$, and (c) $\mathbf{a} - 2\mathbf{b}$.

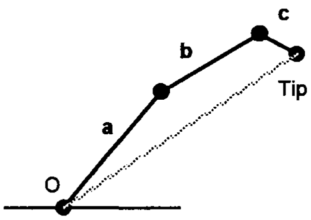


Figure 2.19 *Problem 14*

11 If $\mathbf{a} = 6\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{c} = 5\mathbf{i} - 4\mathbf{j}$, determine: (a) $\mathbf{a} + \mathbf{b} + \mathbf{c}$, (b) $\mathbf{a} - \mathbf{b} - \mathbf{c}$, (c) $\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}$.

12 Determine the magnitude and direction cosines of: (a) $\mathbf{a} = 3\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$, (b) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, (c) $\mathbf{a} = -3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$.

13 The position vectors of points P and Q are $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and $4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ respectively. Determine the length and direction cosines of the vector joining P and Q.

14 For a robot arm involving rigid links connected by flexible joints (Figure 2.19), the link vectors can be represented by $\mathbf{a} = 10\mathbf{i} + 12\mathbf{j} + 1\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + 1\mathbf{j} - 4\mathbf{k}$. Determine the position vector of the tip of the robot from O and the length of each link.

15 Determine the angle made by the vector $\mathbf{v} = -5\mathbf{i} + 12\mathbf{j}$ with the positive sense of the x-axis.

16 A force is specified by the vector $\mathbf{F} = 60\mathbf{i} - 60\mathbf{j} + 30\mathbf{k}$. Calculate the angles made by $\mathbf{F}$ with the x, y and z axes.

2.2 Phasors

Key point

Polar notation is when quantities such as phasors are described by their size and an angle in the form $V\angle\phi$.

Key point

Note that there is a difference between a phasor diagram and a vector diagram. A phasor diagram represents the phasors at one instant of time, a vector diagram represents the vectors without regard to time. Otherwise the mathematics of handling vectors is applicable to phasors.

A convenient way of specifying a phasor is, what is termed, by *polar notation*. Thus a phasor of length V and phase angle ϕ can be represented by $V\angle\phi$. Although the length of a phasor when described in the way shown in Figure 1.25 represents the maximum value of the quantity, it is more usual to specify the length as representing the root-mean-square value. The root-mean-square value is the maximum value divided by $\sqrt{2}$ and so is just a scaled version of the one drawn using the maximum value. This is because in electrical circuit work we are more usually concerned with the root-mean-square current or voltage than the maximum values.

When we are working in the time domain, i.e. the current or voltage is described by a function as time as in $v = V \sin \omega t$, and want to find, say, the sum of two voltages at some instant of time we just add the voltages. Thus, if we have a voltage across one component described by $v_1 = V_1 \sin (\omega t + \phi_1)$ and across a series component by $v_2 = V_2 \sin (\omega t + \phi_2)$, then the sum of the two voltages is:

$$v_1 + v_2 = V_1 \sin (\omega t + \phi_1) + V_2 \sin (\omega t + \phi_2)$$

This equation describes how the voltage sum varies with time.

When we are working with phasors and want to find the phasor representing the sum of two phasors we have to add the phasors in the same way that vector quantities are added. Thus if we have a voltage across one component described by $V_1\angle\phi_1$ and across a series component by $V_2\angle\phi_2$, then the phasor representing the sum of the two voltages is that indicated in Figure 2.20. While we can draw such diagrams for simple situations and obtain the resultant phasor graphically, a more useful technique is to describe a phasor by a complex number and use the techniques for manipulating complex numbers. In the next section we discuss complex numbers and consider their application to electrical circuit analysis in terms of phasors.

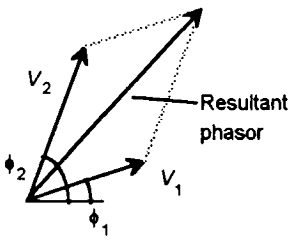


Figure 2.20 *Adding phasors*

2.3 Complex numbers

Key points

Numbers which are multiples of j, where $j = \sqrt{1}$ is termed *imaginary*.

The term *complex number* is used for a combination by addition or subtraction of a real number and a purely imaginary number.

If we square the real number +2 we obtain +4, if we square the real number −2 we obtain +4. Thus the square root of +4 is ±2. But what is the square root of −4? To give an answer we need another form of number. If we invent a number $j = \sqrt{-1}$ (mathematicians often use i rather than j but engineers and scientists generally use j to avoid confusion with i used for current in electrical circuits), then we can write $\sqrt{-4} = \sqrt{-1} \times \sqrt{4} = \pm j2$. Thus the solution of the equation $x + 4 = 0$ is $x = \pm j2$.

The solution of a quadratic equation of the form $ax^2 + bx + c = 0$ is given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus if we want to solve the quadratic equation $x^2 - 4x + 13 = 0$ then:

$$x = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm \sqrt{-9}$$

Key points

$j = \sqrt{-1}$ and thus $j^2 = -1$. Since j^3 can be written as $j^2 \times j$ then $j^3 = -j$. Since j^4 can be written as $j^2 \times j^2$ then $j^4 = +1$.

$j = \sqrt{-1}$, $j^2 = -1$, $j^3 = -j$, $j^4 = +1$

We can represent $\sqrt{-9}$ as $\sqrt{-1} \times \sqrt{+9} = j3$. Thus the solution can be written as $2 \pm j3$, a combination of a real and either plus or minus an imaginary number. Such a pair of roots is known as a *conjugate pair* (see later in this section).

The term complex number is used for the sum of a real number and an imaginary number. Thus a complex number z can be written as $z = a + jb$, where a is the real part of the complex number and b the imaginary part.

Example

Solve the equation $x^2 - 4x + 5 = 0$.

$$x = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm \sqrt{-1} = 2 \pm j$$

The Argand diagram

The effect of multiplying a real number by (−1) is to move the point from one side of the origin to the other. Figure 2.21 illustrates this for (+2) being multiplied by (−1). We can think of the positive number line radiating out from the origin being rotated through 180° to its new position after being multiplied by (−1). But $(-1) = j^2$. Thus, multiplication by j^2 is equivalent to a 180° rotation. Multiplication by j^4 is a multiplication by (+1) and so is equivalent to a rotation through 360°. On this basis it seems reasonable to take a multiplication by j to be equivalent to a rotation through 90° and a multiplication by j^3 a rotation through 270°. This concept of multiplication by j as involving a rotation is the basis of the use of complex numbers to represent phasors in alternating current circuits.

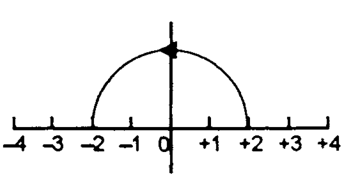


Figure 2.21 *(+2) × (−1)*

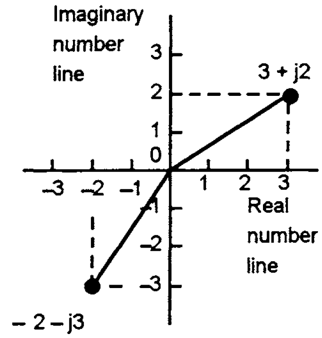


Figure 2.22 *Argand diagram*

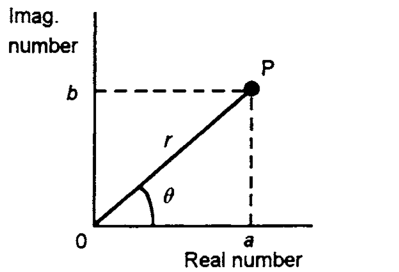


Figure 2.23 *Modulus and argument*

Key points

We can specify a complex number on an Argand diagram in terms of its Cartesian coordinates as $z = a + jb$, or its polar coordinates $z = r\angle\theta$.

The above discussion leads to a diagram, called the *Argand diagram*, which we use to represent complex numbers. Since rotation by 90° from the x-axis on a graph gives the y-axis, the y-axis is used for imaginary numbers and the x-axis for real numbers (Figure 2.22). Figure 2.22 shows how we represent the complex numbers $3 + j2$ and $-2 - j3$ on such a diagram. The line joining the number to the origin is taken as the graphical representation of the complex number.

Modulus and argument

If the complex number $z = a + jb$ is represented on an Argand diagram by the line OP, as in Figure 2.23, then the length r of the line OP is called the *modulus* of the complex number and its inclination θ to the real number axis is termed the *argument* of the complex number. The length of the line is denoted by $|z|$ or modulus z and the argument by θ or arg z.

Using Pythagoras' theorem:

$$|z| = \sqrt{a^2 + b^2} \qquad [13]$$

and, since $\tan\theta = b/a$:

$$\arg z = \tan^{-1}\left(\frac{b}{a}\right) \qquad [14]$$

Since $a = r\cos\theta$ and $b = r\sin\theta$, we can write a complex number z as:

$$z = a + jb = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta) \qquad [15]$$

Thus we can specify a complex number by either stating its location on an Argand diagram in terms of its *Cartesian coordinates* a and b or by specifying the modulus, $|z| = r$, and the argument θ. These are termed its *polar coordinates*. The specification in polar coordinates can be written as:

$$z = |z| \angle \arg z \quad \text{or} \quad z = r\angle\theta \qquad [16]$$

Example

Determine the modulus and argument of the complex number 2 + j2.

$$|z| = \sqrt{a^2 + b^2} = \sqrt{2^2 + 2^2} = 2.8$$

$$\arg z = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{2}{2}\right) = 45°$$

In polar form the complex number could be written as $2.8\angle 45°$.

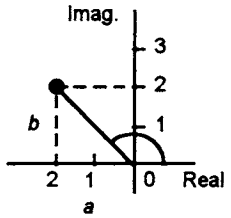


Figure 2.24 *Example*

Example

Write the complex number –2 + j2 in polar form.

$$|z| = \sqrt{a^2+b^2} = \sqrt{(-2)^2+2^2} = 2.8$$

$$\arg z = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{2}{-2}\right)$$

If we sketch an Argand diagram (Figure 2.24) for this complex number we can see that the number is in the second quadrant. The argument is thus –45° + 180° = 135°. In polar form the complex number could be written as 2.8 $\angle$135°.

Example

Write the complex number 10 $\angle$60° in Cartesian form.

$$z = r(\cos\theta + \mathrm{j}\sin\theta) = 10(\cos 60° + \mathrm{j}\sin 60°) = 5 + \mathrm{j}8.7$$

2.3.1 Manipulation of complex numbers

Addition, subtraction, multiplication and division can be carried out on complex numbers in either the Cartesian form or the polar form. Addition and subtraction is easiest when they are in the Cartesian form and multiplication and division easiest when they are in the polar form.

For two complex numbers to be equal, their real parts must be equal and their imaginary parts equal. On an Argand diagram the two numbers then describe the same line. Thus 2 + j3 is *not* equal to 3 + j2 as Figure 2.25 shows.

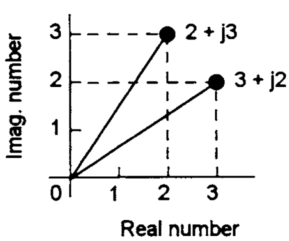


Figure 2.25 *2 + j3 and 3 + j2*

Addition and subtraction

To add complex numbers we add the real parts and add the imaginary parts:

$$(a + \mathrm{j}b) + (c + \mathrm{j}d) = (a + c) + \mathrm{j}(b + d) \qquad [17]$$

On an Argand diagram, this method of adding two complex numbers is the same as the vector addition of two vectors using the parallelogram of vectors, the line representing each complex number being treated as a vector (Figure 2.26).

To subtract complex numbers we subtract the real parts and subtract the imaginary parts:

Key point

To add complex numbers, add the real parts and add the imaginary parts. To subtract complex numbers, subtract the real parts and subtract the imaginary parts.

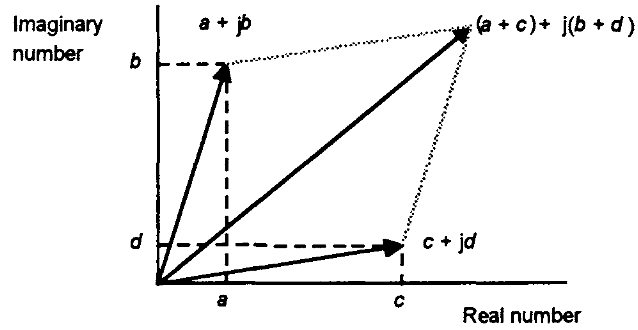


Figure 2.26 *Addition of complex numbers*

$$(a + jb) - (c + jd) = (a - c) + j(b - d) \quad [18]$$

On an Argand diagram, this method of subtracting two complex numbers is the same as the vector subtraction of two vectors. To subtract a vector quantity you reverse its direction and then add it using the parallelogram of vectors (Figure 2.27).

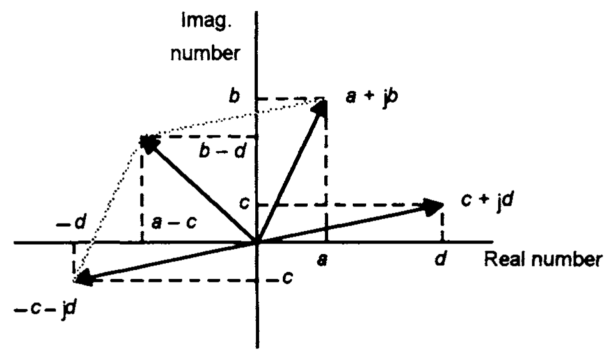


Figure 2.27 *Subtraction of complex numbers*

> **Example**
>
> With $z_1 = 4 + j2$ and $z_2 = 3 + j5$, determine (a) $z_1 + z_2$, (b) $z_1 - z_2$.
>
> (a) $z_1 + z_2 = (4 + 3) + j(2 + 5) = 7 + j7$
>
> (b) $z_1 - z_2 = (4 - 3) + j(2 - 5) = 1 - j3$

Multiplication

Consider the multiplication of the two complex numbers in Cartesian form, $z_1 = a + jb$ and $z_2 = c + jd$. The product z is given by:

$$z = (a + \mathrm{j}b)(c + \mathrm{j}d) = ac + \mathrm{j}(ad + bc) + \mathrm{j}^2bd$$

$$= ac + \mathrm{j}(ad + bc) - bd \qquad [19]$$

Now consider the multiplication of the two complex numbers in polar form, $z_1 = |z_1|\angle\theta_1$ and $z_2 = |z_2|\angle\theta_2$. Using equation [5] we can write:

$$z_1 = |z_1|(\cos\theta_1 + \mathrm{j}\sin\theta_1) \text{ and } z_2 = |z_2|(\cos\theta_2 + \mathrm{j}\sin\theta_2)$$

Thus the product z is given by:

$$z = |z_1|(\cos\theta_1 + \mathrm{j}\sin\theta_1) \times |z_2|(\cos\theta_2 + \mathrm{j}\sin\theta_2)$$

$$= |z_1z_2|\,[\cos\theta_1\cos\theta_2 + \mathrm{j}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) + \mathrm{j}^2\sin\theta_1\sin\theta_2]$$

$$= |z_1z_2|\,[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{j}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

Using the equations for cos $(A + B)$ and sin $(A + B)$, [28] and [29] from Chapter 1:

$$z = |z_1z_2|[\cos(\theta_1 + \theta_2) + \mathrm{j}\sin(\theta_1 + \theta_2)] \qquad [20]$$

Hence we can write for the complex numbers in polar form

$$z = |z_1z_2|\angle(\theta_1 + \theta_2) \qquad [21]$$

Key point

The magnitude of the product of two complex numbers in polar form is the product of the magnitudes of the two numbers and its argument is the sum of the arguments of the two numbers.

Example

Multiply the two complex numbers 2 – j3 and 4 + j1.

$$(2 - \mathrm{j}3)(4 + \mathrm{j}1) = 8 + \mathrm{j}2 - \mathrm{j}12 - \mathrm{j}^2 3$$
$$= 8 + \mathrm{j}2 - \mathrm{j}12 + 3 = 11 - \mathrm{j}10$$

Example

Multiply the two complex numbers $3\angle 40°$ and $2\angle 70°$.

$$3\angle 40° \times 2\angle 70° = (3 \times 2)\angle(40° + 70°) = 6\angle 110°$$

Complex conjugate

If $z = a + \mathrm{j}b$ then the term *complex conjugate* is used for the complex number given by $z^* = a - \mathrm{j}b$. The imaginary part of the complex number changes sign to give the conjugate, conjugates being denoted as z^*. Figure 2.28 shows an Argand diagram with a

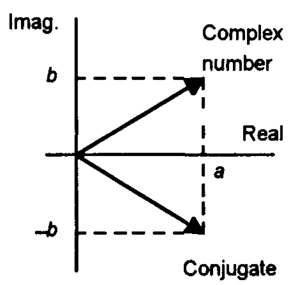


Figure 2.28 *A complex number and its conjugate*

complex number and its conjugate. The complex conjugate is the mirror image of the original complex number.

Consider now the product of a complex number and its conjugate:

$$zz^* = (a + jb)(a - jb) = a^2 - j^2b = a^2 + b^2 \qquad [22]$$

The product of a complex number and its conjugate is a real number.

> **Example**
>
> What is the conjugate of the complex number 2 + j4?
>
> The complex conjugate is 2 – j4.

Division

Consider the division of $z_1 = a + jb$ by $z_2 = c + jd$, i.e.

$$z = \frac{z_1}{z_2} = \frac{a + jb}{c + jd}$$

To divide one complex number by another we have to convert the denominator into a real number. This can be done by multiplying it by its conjugate. Thus:

$$z = \frac{a + jb}{c + jd} \times \frac{c - jd}{c - jd} = \frac{(a + jb)(c - jd)}{c^2 + d^2} \qquad [23]$$

Now consider the division of the two complex numbers when in polar form, $z_1 = |z_1|\angle\theta_1$ and $z_2 = |z_2|\angle\theta_2$:

$$z = \frac{|z_1|(\cos\theta_1 + j\ \sin\theta_1)}{|z_2|(\cos\theta_2 + j\ \sin\theta_2)}$$

Making the denominator into a real number by multiplying it by its conjugate:

$$z = \frac{|z_1|(\cos\theta_1 + j\ \sin\theta_1)}{|z_2|(\cos\theta_2 + j\ \sin\theta_2)} \times \frac{|z_2|(\cos\theta_2 - j\ \sin\theta_2)}{|z_2|(\cos\theta_2 - j\ \sin\theta_2)}$$

$$= \frac{|z_1|}{|z_2|}\left[\frac{(\cos\theta_1 + j\ \sin\theta_1)(\cos\theta_2 - j\ \sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right]$$

But $\cos^2\theta_2 + \sin^2\theta_2 = 1$ (chapter 3, equation [32]) and so:

$$z = \frac{|z_1|}{|z_2|}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$$

Using equations cos $(A - B)$ and sin $(A - B)$, [31] and [29] from Chapter 1:

$$z = \frac{|z_1|}{|z_2|}[\cos(\theta_1 - \theta_2) + \mathrm{j}\sin(\theta_1 - \theta_2)] \quad [24]$$

We can express this as:

$$z = \frac{|z_1|}{|z_2|}\angle(\theta_1 - \theta_2) \quad [25]$$

Key point

To divide two complex numbers in polar form, we divide their magnitudes and subtract their arguments.

Example

Divide 1 + j2 by 1 + j1.

$$\frac{1+\mathrm{j}2}{1+\mathrm{j}1} = \frac{1+\mathrm{j}2}{1+\mathrm{j}1} \times \frac{1-\mathrm{j}1}{1-\mathrm{j}1} = \frac{1+\mathrm{j}1-\mathrm{j}^2 2}{1-\mathrm{j}^2}$$

$$= \frac{3+\mathrm{j}1}{2} = 1.5 + \mathrm{j}0.5$$

Example

Divide $4\angle 40°$ by $2\angle 30°$.

$$\frac{4\angle 40°}{2\angle 30°} = \frac{4}{2}\angle(40° - 30°) = 2\angle 10°$$

2.3.2 Representing phasors by complex numbers

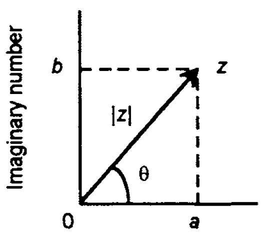


Figure 2.29 *Complex number*

A complex number $z = a + \mathrm{j}b$ can be represented on an Argand diagram by a line (Figure 2.29) of length $|z|$ at an angle θ. Thus we can describe a phasor used to represent, say, a sinusoidal voltage, by a complex number in this Cartesian form as:

$$\mathbf{V} = a + \mathrm{j}b \quad [26]$$

An alternative way of describing a complex number, and hence a phasor, is in polar notation, i.e. the length of the phasor and its angle to some reference axis. Thus we can describe it as:

$$\mathbf{V} = V\angle\theta \quad [27]$$

where V is the magnitude of the phasor and θ its phase angle. The magnitude $|z|$ of a complex number z and its argument θ are given by:

$$|z| = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right) \quad [28]$$

Since $a = |z|\cos\theta$ and $b = |z|\sin\theta$, then:

$$z = |z| \cos \theta + \mathrm{j}|z| \sin \theta = |z|(\cos \theta + \mathrm{j} \sin \theta) \qquad [29]$$

Thus if we have the voltage across one component described by $V\angle\phi$ then we can write this as:

$$\mathbf{V} = V(\cos \phi + \mathrm{j} \sin \phi) \qquad [30]$$

> **Example**
>
> Describe the signal $v = 12 \sin (314t + \pi/4)$ V by a phasor.
>
> The phasor has a magnitude, when expressed as the maximum value, of 12 and argument $\pi/4$. Thus we can describe it as $12\angle\pi/4$ V, or by using equation [29] as $12 \cos \pi/4 + \mathrm{j}12 \sin \pi/4 = 8.49 + \mathrm{j}8.49$ V. If using root-mean-square values then we would have $8.49\angle\pi/4$ r.m.s.V or $6 + \mathrm{j}6$ r.m.s.V.

Adding or subtracting phasors

> **Key point**
>
> Since adding or subtracting complex numbers is easier when they are in Cartesian form rather than polar form, when phasors are to be added or subtracted they should be put in Cartesian form.

If we have the voltage across one component described by $V_1\angle\phi_1$ then we can write: $\mathbf{V}_1 = V_1(\cos \phi_1 + \mathrm{j} \sin \phi_1)$. If we have the voltage across a series component described by $V_2\angle\phi_2$, then: $\mathbf{V}_2 = V_2(\cos \phi_2 + \mathrm{j} \sin \phi_2)$. The phasor for the sum of the two voltages is then obtained by adding the two complex numbers. Thus:

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = V_1(\cos \phi_1 + \mathrm{j} \sin \phi_1) + V_2(\cos \phi_2 + \mathrm{j} \sin \phi_2)$$

$$= (V_1 \cos \phi_1 + V_2 \cos \phi_2) + \mathrm{j}(V_1 \sin \phi_1 + V_2 \sin \phi_2) \qquad [31]$$

Subtraction is carried out in a similar manner. Since adding or subtracting complex numbers is easier when they are in Cartesian form rather than polar form, when phasors are to be added or subtracted they should be put in Cartesian form.

> **Example**
>
> A circuit has three components in series. If the voltages across each component are described by phasors 4 V, j2 V and 3 + j4 V, what is the voltage phasor describing the voltage across the three components?
>
> Since the components are in series, the resultant phasor voltage is described by the phasor:
>
> $\mathbf{V} = 4 + \mathrm{j}2 + 3 + \mathrm{j}4 = 7 + \mathrm{j}6$

Example

A circuit has two components in series. If the voltages across each component are described by phasors $4\angle 60°$ V and $2\angle 30°$ V, what is the voltage phasor describing the voltage across the two components?

For adding complex numbers it is simplest to convert the phasors into Cartesian notation. Thus:

$$\mathbf{V} = (4 \cos 60° + j4 \sin 60°) + (2 \cos 30° + j2 \sin 30°)$$

$$= 2 + j3.46 + 1.73 + j1 = 3.73 + j4.46 \text{ V}$$

If we want this phasor in polar notation then:

$$V = \sqrt{3.73^2 + 4.46^2} = 5.81$$

$$\phi = \tan^{-1}\frac{4.46}{3.73} = 50°$$

Thus the phasor is $5.81\angle 50°$ V.

Key point

Multiplication or division of complex numbers is easiest when they are in polar form.

Multiplication or division of phasors

Multiplication or division of complex numbers can be carried out when they are in either Cartesian form or polar form, being easiest when they are in polar form. Thus, if we have a voltage across a component described by $\mathbf{V} = V\angle\phi$ and the current by $\mathbf{I} = I\angle\theta$ then the product of the two phasors is:

$$\mathbf{VI} = VI\angle(\phi + \theta) \qquad [32]$$

If the voltage and current were in Cartesian form, i.e. in the form $\mathbf{V} = a + jb$ and $\mathbf{I} = c + jd$ then the product is:

$$\mathbf{VI} = (a + jb)(c + jd) = (ac - bd) + j(bc + ad) \qquad [33]$$

For division, if we have a voltage across a component described by $\mathbf{V} = V\angle\phi$ and the current by $\mathbf{I} = I\angle\theta$ then:

$$\frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\phi}{I\angle\theta} \qquad [34]$$

If the voltage and current were in Cartesian form, i.e. in the form $\mathbf{V} = a + jb$ and $\mathbf{I} = c + jd$ then:

$$\frac{\mathbf{V}}{\mathbf{I}} = \frac{a + jb}{c + jd} = \frac{a + jb}{c + jd} \times \frac{c - jd}{c - jd} = \frac{(a + jb)(c - jd)}{c^2 - d^2} \qquad [35]$$

Example

If phasor **V** is represented by $10\angle 30°$ and **I** by $2\angle 45°$, determine **VI** and **V/I**.

$$\mathbf{VI} = (10 \times 2)\angle(30° + 45°) = 20\angle 75°$$

$$\frac{\mathbf{V}}{\mathbf{I}} = \frac{10\angle 30°}{2\angle 45°} = 5\angle(-15°)$$

Kirchhoff's laws and phasors

Kirchhoff's laws apply to the voltages and currents in a circuit at any instant of time. Thus the voltage law that the sum of the voltages taken round a closed loop is zero means that, with alternating voltages having values of v_1, v_2, v_3, etc. at the same instant of time:

$$v_1 + v_2 + v_3 + \ldots = 0$$

and so, if these voltages are sinusoidal:

$$V_1 \sin(\omega t + \phi_1) + V_2 \sin(\omega t + \phi_2) + V_3 \sin(\omega t + \phi_3) + \ldots = 0$$

We can consider each of these sinusoidal voltages to be the vertical projection of the phasor describing it. Thus we must have:

$$\mathbf{V_1} + \mathbf{V_2} + \mathbf{V_3} + \ldots = 0$$

Kirchhoff's voltage law can thus be stated as: *the sum of the phasors of all the voltages around a closed loop is zero.* Kirchhoff's current law can be stated as the sum of all the currents at a node is zero, i.e. the current entering a junction equals the current leaving it. In a similar way we can state this law for sinusoidal currents as: *the sum of the phasors of the currents at a node is zero, i.e. the sum of the phasors for currents entering a junction equals that for those leaving it.*

Example

A circuit has two components in parallel. If the currents through the components can be described by the phasors 2 + j4 A and 4 + j1 A, what is the phasor describing the current entering the junction?

Using Kirchhoff's current law we must have: the phasor for current entering junction = phasor sum for currents leaving the junction. Hence:

phasor for current entering = 2 + j4 + 4 + j1 = 6 + j5 A

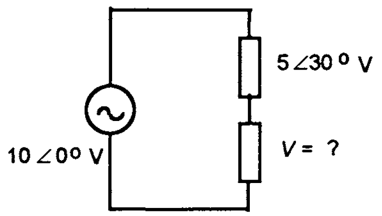


Figure 2.30 *Example*

Example

For the a.c. circuit shown in Figure 2.30, determine the unknown voltage.

Using Kirchhoff's voltage law, and writing the phasors in Cartesian notation:

$10 + j0 = (5\cos 30° + j5\sin 30°) + \mathbf{V}$

Thus:

$\mathbf{V} = 10 - 4.33 + j2.5 = 5.67 - j2.5$ V

or in polar notation:

$\mathbf{V} = \sqrt{5.67^2 + 2.5^2}\ \angle\ \tan^{-1}(-2.5/5.67 = 6.2\angle 336.2°$ V

Key point

Impedance is described by a complex number but is not a phasor since it does not describe a sinusoidally varying quantity. It describes a line on an Argand diagram but not one that rotates with an angular velocity. Hence, in this book bold print is not used for it. In some textbooks, however, it is written in bold print because it is complex.

Impedance

The term *impedance* Z is defined as the ratio of the phasor voltage across a component to the phasor current through it:

$$Z = \frac{\mathbf{V}}{\mathbf{I}} \qquad [36]$$

Thus if we have $\mathbf{V} = V\angle\theta$ and $\mathbf{I} = I\angle\phi$ then:

$$Z = \frac{V\angle\theta}{I\angle\phi} = \frac{V}{I}\angle(\theta - \phi)$$

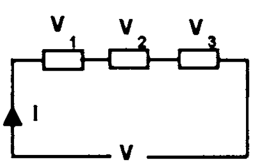


Figure 2.31 *Impedances in series*

If we have impedances connected in series (Figure 2.31), then Kirchhoff's voltage law gives:

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3$$

Dividing by the phasor current, the current being the same through each:

$$\frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_1}{\mathbf{I}} + \frac{\mathbf{V}_2}{\mathbf{I}} + \frac{\mathbf{V}_3}{\mathbf{I}}$$

Hence the total impedance Z is the sum of the impedances of the three impedances:

$$Z = Z_1 + Z_2 + Z_3 \qquad [37]$$

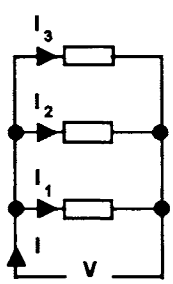


Figure 2.32 *Impedances in parallel*

Consider the parallel connection of impedances (Figure 2.32). Kirchhoff's current law gives:

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3$$

Dividing by the phasor voltage, the voltage being the same for each impedance:

$$\frac{\mathbf{I}}{\mathbf{V}} = \frac{\mathbf{I}_1}{\mathbf{V}} + \frac{\mathbf{I}_2}{\mathbf{V}} + \frac{\mathbf{I}_3}{\mathbf{V}}$$

Thus the total impedance Z is given by:

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \qquad [38]$$

Example

If the voltage across a component is 4 sin ωt V and the current through it 2 sin(ωt – 30°) A, what is its impedance?

Using equation [36] with the phasors in polar notation:

$$Z = \frac{4\angle 0°}{2\angle(-30°)} = 2\angle 30°\ \Omega$$

or Z = 2 cos 30° + 2 sin 30° = 1.73 + j1 Ω.

Example

What is the total impedance of a circuit with impedances of 2 + j5 Ω, 1 – j3 Ω and 4 + j1 Ω in series?

Z = 2 + j5 + 1 – j3 + 4 + j1 = 7 + j3 Ω

Example

What is the total impedance of impedances 4∠30° Ω in parallel with 2∠(–20°) Ω.

$$\frac{1}{Z} = \frac{1}{4\angle 30°} + \frac{1}{2\angle(-20°)} = 0.25\angle(-30°) + 0.5\angle 20°$$

$$= 0.25\cos(-30°) + \mathrm{j}0.25\sin 30° + 0.5\cos 20° + \mathrm{j}0.5\sin 20°$$

$$= 0.686 + \mathrm{j}0.296$$

$$= \sqrt{0.686^2 + 0.296^2}\ \angle\tan^{-1}(0.296/0.686)$$

$$= 0.747\angle 23.3°$$

Hence Z = 1.339∠(–23.3°) Ω.

Circuit elements

For a *pure resistor* the current through it is in phase with the voltage across it. Thus for a voltage phasor of $V\angle 0°$ we must have a current phasor of $I\angle 0°$ and so the impedance of the circuit element is:

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle 0°}{I\angle 0°} = \frac{V}{I}\angle 0°$$

The impedance is the real number V/I which is the resistance R.

For a *pure capacitance* the current leads the voltage by 90°. Thus for a voltage phasor of $V\angle 0°$ we must have a current phasor of $I\angle 90°$ and so the impedance of the circuit element is:

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle 0°}{I\angle 90°} = \frac{V}{I}\angle(-90°)$$

The impedance is thus $-\mathrm{j}(V/I)$ and is just an imaginary quantity. The term *capacitive reactance* X_C is used for the ratio of the maximum, or r.m.s., voltage and current and thus for a pure capacitance:

$$Z = -\mathrm{j}X_C \qquad [39]$$

For a *pure inductance* the current lags the voltage by 90°. Thus for a voltage phasor of $V\angle 0°$ we must have a current phasor of $I\angle(-90°)$ and so the impedance of the circuit element is:

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle 0°}{I\angle(-90°)} = \frac{V}{I}\angle 90°$$

The impedance is thus $\mathrm{j}(V/I)$ and is just an imaginary quantity. The term *inductive reactance* X_L is used for the ratio of the maximum, or r.m.s., voltage and current and thus for a pure inductance:

$$Z = \mathrm{j}X_L \qquad [40]$$

Example

Determine the impedance of a 100 Ω resistance in series with a capacitive reactance of 5 Ω.

$Z = R - \mathrm{j}X_C = 100 - \mathrm{j}5\ \Omega$

Example

Express in Cartesian and polar notation, the impedance of each of the following circuits at a frequency of 50 Hz:

(a) a resistance of 20 Ω in series with an inductance of 0.1 H,
(b) a resistance of 50 Ω in series with a capacitance of 40 μF.

Also calculate the size of the current in each case and its phase relative to an applied voltage of 230 V at 50 Hz.

(a) With 50 Hz we have $\omega = 2\pi f = 2\pi \times 50 = 314.16$ rad/s. Thus:

$Z = R + jX_L = 20 + j314.16(0.1) = 20 + j31.42\ \Omega$

Converting this to polar notation gives $|Z| = \sqrt{(20^2 + 31.42^2)} = 37.25\ \Omega$. The phase is $\tan^{-1}(X_L/R) = \tan^{-1}(31.42/20) = 57.52°$ and so the impedance is $37.25\angle 57.52°\ \Omega$.

The current **I** = **V**/*Z* and so is:

$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{230\angle 0°}{37.25\angle 57.52°} = 6.17\angle -57.52°\ \text{A}$$

(b) The capacitance reactance $X_C = 1/\omega C$ and so:

$$Z = R - jX_C = 50 - j\frac{1}{314.16 \times 40 \times 10^{-6}} = 50 - j79.58\ \Omega$$

Converting this to polar notation $|Z| = \sqrt{(50^2 + 79.58^2)} = 93.98\ \Omega$. The phase is $\tan^{-1}(X_C/R) = \tan^{-1}(-79.58/50) = -57.85°$ and so the impedance is $93.98\angle -57.85°\ \Omega$.

The current **I** = **V**/*Z* and so is:

$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{230\angle 0°}{93.98\angle -57.85°} = 2.45\angle 57.85°\ \text{A}$$

Example

Calculate the resistance and the series inductance or capacitance for each of the following impedances if the frequency is 50 Hz: (a) $Z = 10 + j15\ \Omega$, (b) $Z = -j80\ \Omega$, (c) $Z = 50\angle 30°\ \Omega$, (d) $Z = 120\angle -60°\ \Omega$.

$\omega = 2\pi f = 2\pi \times 50 = 314$ rad/s

(a) Comparing this with $Z = R + j\omega L$, then $R = 10\ \Omega$ and $X_L = 15\ \Omega$. Since $X_L = \omega L$ = then $L = 15/314 = 0.048$ H.
(b) Here $R = 0$ and the capacitive reactance $X_C = 80\ \Omega$. Since $X_C = 1/\omega C$ then $C = 1/(314 \times 80) = 39.8 \times 10^{-6}$ F or 39.8 μF.

(c) This gives in Cartesian notation (see equation [29]) $Z = 50 \cos 30° + j50 \sin 30° = 43.3 + j25\ \Omega$. We can compare this with $Z = R + jX_L$ and so $R = 43.3\ \Omega$ and $X_L = 25\ \Omega$. Since $X_L = \omega L$ then $L = 25/314 = 0.080$ H.
(d) This gives in Cartesian notation (see equation [29]) $Z = 120 \cos -60° + j120 \sin -60° = 60 - j104\ \Omega$. We can compare this with $Z = R - jX_C$ and so $R = 60\ \Omega$ and $X_C = 104\ \Omega$. Since $X_C = 1/\omega C$ then $C = 1/(314 \times 104) = 30.7\ \mu F$.

Problems 2.3

1 Simplify (a) j^7, (b) j^8, (c) $j^2 \times j$, (d) j^5/j^3.

2 Solve the following equations:

(a) $x^2 + 16 = 0$, (b) $x^2 + 4x - 5 = 0$, (c) $2x^2 - 2x + 3 = 0$

3 Express the following complex numbers in polar form:

(a) $-4 + j$, (b) $-3 - j4$, (c) 3, (d) $-j6$, (e) $1 + j$, (f) $3 - j2$

4 Express the following complex numbers in Cartesian form:

(a) $5 \angle 120°$, (b) $10\angle 45°$, (c) $6\angle 180°$, (d) $2.8\angle 76°$,

(e) $2(\cos 30° + j \sin 30°)$, (f) $3(\cos 60° - j \sin 60°)$

5 If $z_1 = 3 + j2$ and $z_2 = -2 + j4$, determine the values of:

(a) $z_1 + z_2$, (b) $z_1 - z_2$, (c) $z_1 z_2$, (d) $\frac{1}{z_1}$, (e) $\frac{z_1}{z_2}$

6 Evaluate the following:

(a) $(2 + j3) + (3 - j5)$, (b) $(-4 - j6) + (2 + j5)$,

(c) $(2 + j2) - (3 - j5)$, (d) $(2 + j4) - (1 + j4)$, (e) $4(3 + j2)$,

(f) $j2(3 + j5)$, (g) $(1 - j2)(3 + j4)$,

(h) $(2 + j2)(3 - j3)$, (i) $(1 + j2)(4 - j3)$,

(j) $\frac{6 + j3}{4 - j2}$, (k) $\frac{1}{3 + j2}$, (l) $\frac{1 + j1}{1 - j1}$, (m) $\frac{3 + j2}{1 - j3}$

7 If $z_1 = 10\angle 20°$, $z_2 = 2\angle 40°$ and $z_3 = 5\angle 60°$, evaluate the following:

(a) $z_1 z_2$, (b) $z_1 z_3$, (c) $\frac{1}{z_1}$, (d) $\frac{1}{z_2}$, (e) $\frac{z_1}{z_2}$, (f) $\frac{z_2}{z_3}$

8 Describe the following signals by phasors written in both polar and Cartesian forms, taking the magnitude to represent the maximum value:

(a) $10 \sin (2\pi 50t - \pi/6)$, (b) $10 \sin (314t + 150°)$,

(c) $22 \sin (628t + \pi/4)$

9 Determine, in both Cartesian and polar forms, the sum of the following phasors:

(a) $4\angle 0°$ and $3\angle 60°$, (b) $2 + j3$ and $-4 + j4$,
(c) $4\angle \pi/3$ and $2\angle \pi/6$

10 If phasors **A**, **B** and **C** are represented by $\mathbf{A} = 10\angle 30°$, $\mathbf{B} = 2.5\angle 60°$ and $\mathbf{C} = 2\angle 45°$ determine:

(a) **AB**, (b) **AC**, (c) **A**(**B** + **C**), (d) **A**/**B**, (e) **B**/**C**,
(f) **C**/(**A** + **B**)

11 If $v_1 = 10 \sin \omega t$ and $v_2 = 20 \sin (\omega t + 60°)$, what is (a) the phasor describing the sum of the two voltages and (b) its time-domain equation?

12 If the voltage across a component is $5 \sin (314t + \pi/6)$ V and the current through it $0.2 \sin (314t + \pi/3)$ A, what is its impedance?

13 A voltage of 100 V is applied across a circuit of impedance $40 + j30\ \Omega$, what is, in polar notation, the current taken?

14 Determine, in Cartesian form, the total impedances of:

(a) $10\ \Omega$ in series with $2 - j5\ \Omega$,
(b) $100\angle 30°\ \Omega$ in series with $100\angle 60°\ \Omega$,
(c) $20\angle 30°\ \Omega$ in series with $15\angle(-10°)\ \Omega$,
(d) $20\angle 30°\ \Omega$ in parallel with $6\angle(-90°)\ \Omega$,
(e) $10\ \Omega$ in parallel with $-j2\ \Omega$,
(f) $j40\ \Omega$ in parallel with $j20\ \Omega$

15 Determine, in Cartesian form, the impedance of:

(a) a resistance of 5 Ω in series with an inductive reactance of 2 Ω,
(b) a resistance of 50 Ω in series with a capacitive reactance of 10 Ω,
(c) a resistance of 2 Ω in series with an inductive reactance of 5 Ω and a capacitive reactance of 4 Ω,
(d) three elements in parallel, a resistance of 2 Ω, an inductive reactance of 10 Ω and a capacitive reactance of 5 Ω,
(e) an inductive reactance of 500 Ω in parallel with a capacitive reactance of 100 Ω

3 Mathematical models

Summary

Engineers frequently have to devise and use mathematical models for systems. Mathematical modelling is the activity by which a problem involving the real-world is translated into mathematics to form a model which can then be used to provide information about the original real problem. Such mathematical models are essential in the design-to-test phase, in particular forming the benchmark by which a computer generated simulation can be measured prior to manufacturing a prototype. This chapter is an introduction to mathematical modelling for engineering systems, later chapters involving more detailed consideration of models.

Objectives

By the end of this chapter, the reader should be able to:

- understand what is meant by a mathematical model and how such models are formulated;
- devise mathematical models for simple systems.

3.1 Modelling

Consider some real problems:

- Tall buildings are deflected by strong winds, can we devise a model which can be used to predict the amount of deflection of a building for particular wind strengths?
- Cars have suspension systems, can we devise a model which can be used to predict how a car will react when driven over a hump in the road?
- When a voltage is connected to a d.c. electrical motor, can we devise a model which will predict how the torque developed by the motor will depend on the voltage?
- Can we devise a model to enable the optimum shaft to be designed for a power transmission system connecting a motor to a load?

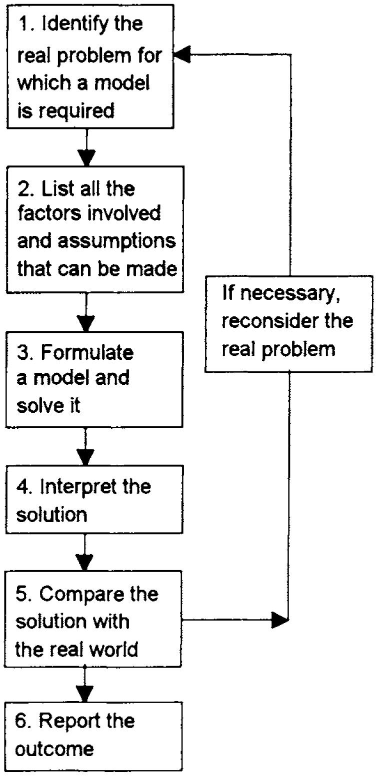


Figure 3.1 *The processes involved in devising a model*

Key point

With reference to Figure 3.1 and point 5, together with any reconsideration of the problem, quite often this involves the formulation of a computer model. The aim is to use the computer model to 'speed' up the analysis of the mathematical model in order to compare the results with collected data from the real world. Such software models therefore have to be benchmarked against the mathematical model and real world data to be validated prior to being used.

- Can we design a model to enable an appropriate transducer to be selected as part of the monitoring/activation circuit for the safe release of an air bag in a motor vehicle under crash conditions?
- Can we design models which will enable failure to be predicted in static and dynamic systems?
- Can we design models to enable automated, robotic controlled systems to be designed?

Such questions as those above are encountered by design and manufacturing system engineers daily. Quite often it is the accuracy of a mathematical model that will determine the success or otherwise of a new design. Such models also greatly help to reduce the design-test-evaluation-manufacture process lead time.

3.1.1 How do we devise models?

The tactics adopted to devise models involve a number of stages which can be summarised by the block diagram of Figure 3.1. The first stage involves identifying what the real problem is and then identifying what factors are important and what assumptions can be made. These assumptions are used in order to simplify the model and enable an initial model to be formulated which we can check as being a reasonable approximation. Only when we are confident with the model do we build in further considerations to make the models even more accurate! For example, when modelling mechanical systems, we often initially ignore friction and the consequential heat generation; however, as the model is refined we have to consider such effects and adjust the initial 'ball park' model in order for its application to the real world to be valid. This stage will generally involve collecting data. The next stage is then to formulate a model. An essential part of this is to translate verbal statements into mathematical relationships. When solutions are then produced from the model, they need to be compared with the real world and, if necessary the entire cycle repeated.

Example of formulating a model

As an illustration, consider how we might approach the problem we started this section with:

> Tall buildings are deflected by strong winds, can we devise a model which can be used to predict the amount of deflection of a building for particular wind strengths?

The simplest form we might consider is that of a tall building which is subject to wind pressure over its entire height, the building being anchored at the ground but free to deflect at its top (Figure 3.2). If we assume that the wind pressure gives a uniform loading over the entire height of the building and does not

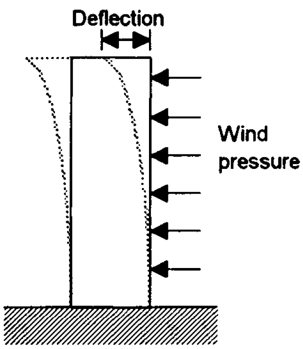


Figure 3.2 *The building deflection problem*

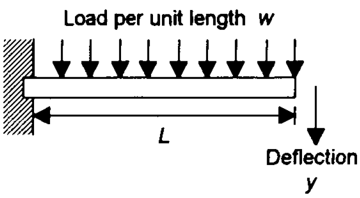


Figure 3.3 *Deflection of a uniformly loaded cantilever*

fluctuate, then we might consider the situation is rather like the deflection of the free end of a cantilever when subject to a uniformly distributed load (Figure 3.3). For such a beam the deflection y is given by:

$$y = \frac{wL^4}{8EI} \qquad [1]$$

where L is the length of the beam, w the load per unit length, E the modulus of elasticity and I the second moment of area. The modulus of elasticity if a measure of the stiffness of the material and the second moment of area for a rectangular section is $bd^3/12$, with b being the breadth and d the depth. This would suggest that a stiff structure would deflect less and also one with a large cross-section would deflect less. Hence we might propose a model of the form:

$$\text{deflection} \propto \frac{pH^4}{\text{stiffness} \times bd^3} \qquad [2]$$

with p being the wind pressure and related to the wind velocity, H the height of the building, b its breadth and d its depth. Thus, a short, squat building will be less deflected than a tall slender one of the same building materials.

3.1.2 Lumped element modelling

Often in engineering we can devise a model for a system by considering it to be composed of a number of basic elements. We consider the characteristics of the behaviour of the system and 'lump' all the similar behaviour characteristics together and represent them by a simple element. For some elements, the relationship between their input and output is a simple proportionality, in other cases it involves a rate of change with time or even the rate of change with time of a rate of change with time.

> **Key point**
>
> Lumped models are devised by considering each of the basic behaviour characteristics of a system and representing them by a single element.

Mechanical systems

Mechanical systems can be considered to be made up of three basic elements which represent the stiffness, damping and inertia of the system:

- ***Spring element***
  The 'springiness' or 'stiffness' of a system can be represented by a spring (Figure 3.4(a)). The force F is proportional to the extension x of the spring:

  $$F = kx \qquad [3]$$

- ***Damper element***
  The 'damping' of a mechanical system can be represented by a dashpot. This is a piston moving in a viscous medium in a

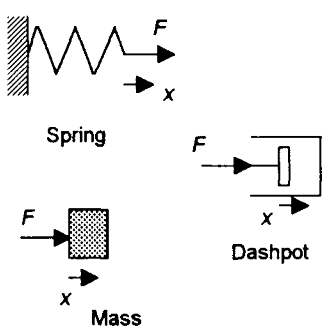


Figure 3.4 *Mechanical system building blocks*

cylinder (Figure 3.4(b)). The damping force F is proportional to the velocity v of the damping element:

$$F = cv$$

where c is a constant. Since the velocity is equal to the rate of change of displacement x:

$$F = c\frac{dx}{dt} \qquad [4]$$

- ***Mass or inertia element***
  The 'inertia' of a system, i.e. how much it resists being accelerated can be represented by mass m. Since the force F acting on a mass is related to its acceleration a by $F = ma$ and acceleration is the rate of change of velocity, with velocity being the rate of change of displacement x:

$$F = ma = m\frac{dv}{dt} = m\frac{d}{dt}\left(\frac{dx}{dt}\right) = m\frac{d^2x}{dt^2} \qquad [5]$$

To develop the equations relating inputs and outputs we use Newton's laws of motion.

Example

Develop a lumped-model for a car suspension system which can be used to predict how a car will react when driven over a hump in the road (the second problem listed earlier in this chapter).

Such a system can have its suspension represented by a spring, the shock absorbers by a damper and the mass of the car and its passengers by a mass. The model thus looks like Figure 3.5(a). We can then devise an equation showing how the output of the system, namely the displacement of the passengers with time depends on the input of the displacement of a car wheel as it rides over the road surface.

This is done by considering a *free-body diagram*, this being a diagram of the mass showing just the external forces acting on it (Figure 3.5(b)). We can then relate the net force acting on the mass to its acceleration by the use of Newton's law, hence obtaining an equation which relates the input to the output.

The relative extension of the spring is the difference between the displacement x of the mass and the input displacement y. Thus, the force due to the spring F_s is $k(x - y)$. The dashpot force F_d is $c(dx/dt - dy/dt)$. Hence, applying Newton's law:

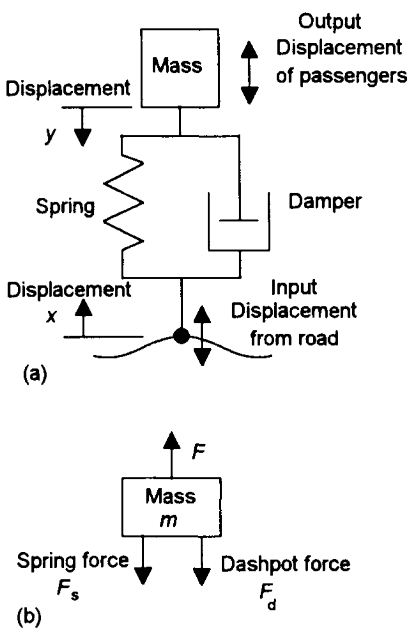


Figure 3.5 *(a) Model for the car suspension system, (b) free-body diagram*

$F - F_s - F_d = ma$

and so we can write:

$$m\frac{d^2x}{dt^2} = F - k(x-y) - c\left(\frac{dx}{dt} - \frac{dy}{dt}\right)$$

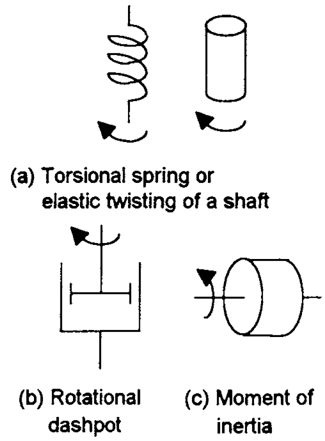


Figure 3.6 *Rotational system elements: (a) torsional spring, (b) rotational dashpot, (c) moment of inertia*

Rotational systems

For rotational systems, e.g. the drive shaft of a motor, the basic building blocks are a torsion spring, a rotary damper and the moment of inertia (Figure 3.6).

- ***Torsional spring***
  The 'springiness' or 'stiffness' of a rotational spring is represented by a torsional spring. The torque T is proportional to the angle θ rotated:

  $$T = k\theta \qquad [6]$$

  where k is a constant.

- ***Rotational dashpot***
  The damping inherent in rotational motion is represented by a rotational dashpot. The resistive torque T is proportional to the angular velocity ω and thus, since ω is the rate of change of angle θ with time:

  $$T = c\omega = c\frac{d\theta}{dt} \qquad [7]$$

  where c is a constant.

- ***Inertia***
  The inertia of a rotational system is represented by the moment of inertia I of a mass. The torque T needed to produce an acceleration a is given by $T = Ia$ and thus, since a is the rate of change of angular velocity ω with time and angular velocity is the rate of change of angle θ with time:

  $$T = Ia = I\frac{d\omega}{dt} = I\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = I\frac{d^2\theta}{dt^2} \qquad [8]$$

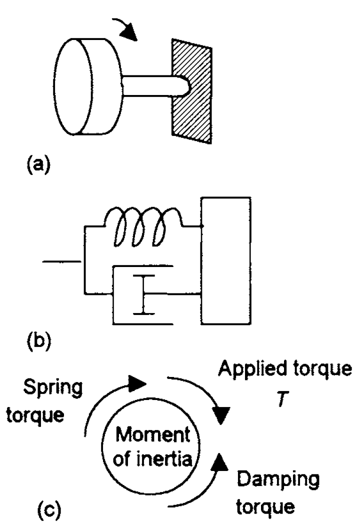


Figure 3.7 *Example*

Example

Represent as a lumped-model the system shown in Figure 3.7(a) of the rotation of a disk as a result of twisting a shaft.

Figure 3.7(b) shows the lumped-model and Figure 3.7(c) the free-body diagram for the system.

The torques acting on the disk are the applied torque, the spring torque and the damping torque. The torque due to the spring is $k\theta$ and the damping torque is $c(\mathrm{d}\theta/\mathrm{d}t)$. Hence, since the net torque acting on the mass is:

net torque = T – spring torque – damping torque

we have:

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = T - k\theta - c\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Electrical systems

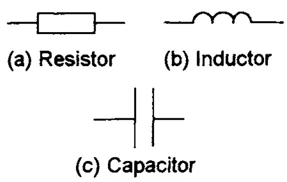


Figure 3.8 *Electrical system elements*

The basic elements of electrical systems are the resistor, inductor and capacitor (Figure 3.8).

- ***Resistor***
  The resistor represents the electrical resistance of the system. The potential difference v across a resistor is proportional to the current i through it:

$$v = Ri \qquad [9]$$

- ***Inductor***
  The inductor represents the electrical inductance of the system. For an inductor, the potential difference v across it depends on the rate of change of current i through it and we can write:

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t} \qquad [10]$$

- ***Capacitor***
  The capacitor represents the electrical capacitance of the system. For a capacitor, the charge q on the capacitor plates is related to the voltage v across the capacitor by $q = Cv$, where C is the capacitance. Since current i is the rate of movement of charge:

$$i = \frac{\mathrm{d}q}{\mathrm{d}t}$$

  But $q = Cv$, with C being constant, so:

$$i = \frac{\mathrm{d}(Cv)}{\mathrm{d}t} = C\frac{\mathrm{d}v}{\mathrm{d}t} \qquad [11]$$

To develop the models for systems which we describe by electrical circuits involving resistance, inductance and capacitance we use Kirchhoff's laws.

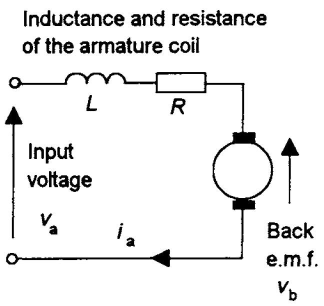


Figure 3.9 *Lumped-model for a d.c. motor*

Example

Develop a lumped-system model for a d.c. motor relating the current through the armature to the applied voltage.

The motor consists basically of the armature coil, this being free to rotate, which is located in the magnetic field provided by either a permanent magnet or a current through field coils. When a current flows through the armature coil, forces acting on as a result of the current carrying conductors being in a magnetic field. As a result, the armature coil rotates. Since the armature is a coil rotating in a magnetic field, a voltage is induced in it in such a direction as to oppose the change producing it, i.e. there is a back e.m.f. Thus the electrical circuit we can use to describe the motor has two sources of e.m.f., that applied to produce the armature current and the back e.m.f. If we consider a motor where there is either a permanent magnet or separately excited field coils, then the lumped electrical circuit model is as shown in Figure 3.9. We can consider there are just two elements, an inductor and a resistor, to represent the armature coil. The equation is thus:

$$v_a - v_b = Ri_a + L\frac{di_a}{dt}$$

This can be considered to be the first stage in addressing the problem posed earlier in the chapter: When a voltage is connected to a d.c. electrical motor, can we devise a model which will predict how the torque developed by the motor will depend on the voltage? The torque generated will be proportional to the current through the armature.

Thermal systems

Thermal systems have two basic building blocks with thermal systems, resistance and capacitance.

- ***Thermal resistance***

  The thermal resistance R is the resistance offered to the rate of flow of heat q and is defined by:

$$q = \frac{T_1 - T_2}{R} \qquad [12]$$

where $T_1 - T_2$ is the temperature difference through which the heat flows.

For heat conduction through a solid we have the rate of flow of heat proportional to the cross-sectional area A and the temperature gradient. Thus, for two points at temperatures T_1 and T_2 and a distance L apart, we can write:

$$q = Ak\frac{T_1 - T_2}{L} \quad [13]$$

with k being the thermal conductivity. With this mode of heat transfer, the thermal resistance R is L/Ak.

For heat transfer by convection between two points, Newton's law of cooling gives:

$$q = Ah(T_2 - T_1) \quad [14]$$

where $(T_2 - T_1)$ is the temperature difference, h the coefficient of heat transfer and A the surface area across which the temperature difference is. The thermal resistance with this mode of heat transfer is thus $1/Ah$.

- ***Thermal capacitance***

The thermal capacitance is a measure of the store of internal energy in a system. If the rate of flow of heat into a system is q_1 and the rate of flow out q_2 then the rate of change of internal energy of the system is $q_1 - q_2$. An increase in internal energy can result in a change in temperature:

change in internal energy = mc × change in temperature

where m is the mass and c the specific heat capacity. Thus the rate of change of internal energy is equal to mc times the rate of change of temperature. Hence:

$$q_1 - q_2 = mc\frac{dT}{dt} \quad [15]$$

This equation can be written as:

$$q_1 - q_2 = C\frac{dT}{dt} \quad [16]$$

where the capacitance $C = mc$.

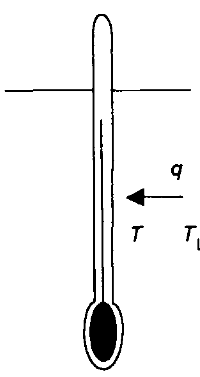


Figure 3.10 *Example*

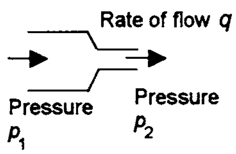
Example

Develop a lumped-model for the simple thermal system of a thermometer at temperature T being used to measure the temperature of a liquid when it suddenly changes to the higher temperature of T_L (Figure 3.10).

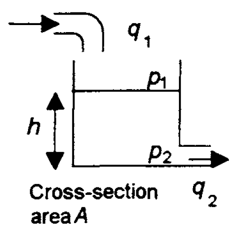
When the temperature changes there is heat flow q from the liquid to the thermometer. If R is the thermal resistance to heat flow from the liquid to the thermometer then $q = (T_L - T)/R$. Since there is only a net flow of heat from the liquid to the thermometer, if the thermal capacitance of the thermometer is C, then $q = C\,dT/dt$.

Thus, the equation for the model is:

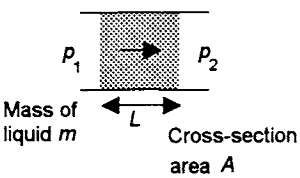
$$C\frac{dT}{dt} = \frac{T_L - T}{R}$$



(a) Resistance



(b) Capacitance



(c) Inertance

Figure 3.11 *Hydraulic system elements*

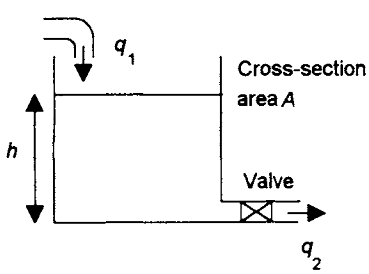


Figure 3.12 *Example*

Hydraulic systems

For a fluid system the three building blocks are resistance, capacitance and inertance. Hydraulic fluid systems are assumed to involve an incompressible liquid; pneumatic systems, however, involve compressible gases and consequently there will be density changes when the pressure changes. Here we will just consider the simpler case of hydraulic systems. Figure 3.11 shows the basic form of building blocks for hydraulic systems.

- ***Hydraulic resistance***
  Hydraulic resistance R is the resistance to flow which occurs when a liquid flows from one diameter pipe to another (Figure 3.11(a)) and is defined as being given by the hydraulic equivalent of Ohm's law:

$$p_1 - p_2 = Rq \qquad [17]$$

- ***Hydraulic capacitance***
  Hydraulic capacitance C is the term used to describe energy storage where the hydraulic liquid is stored in the form of potential energy (Figure 3.11(b)). The rate of change of volume V of liquid stored is equal to the difference between the volumetric rate at which liquid enters the container q_1 and the rate at which it leaves q_2, i.e.

$$q_1 - q_2 = \frac{dV}{dt}$$

But $V = Ah$ and so:

$$q_1 - q_2 = A\frac{dh}{dt}$$

The pressure difference between the input and output is:

$$p_1 - p_2 = p = h\rho g$$

Hence, substituting for h gives:

$$q_1 - q_2 = \frac{A}{\rho g}\frac{dp}{dt} \qquad [18]$$

The hydraulic capacitance C is defined as:

$$C = \frac{A}{\rho g} \qquad [19]$$

and thus we can write:

$$q_1 - q_2 = C\frac{dp}{dt} \qquad [20]$$

- ***Hydraulic inertance***
  Hydraulic inertance is the equivalent of inductance in electrical systems. To accelerate a fluid a net force is required and this is provided by the pressure difference (Figure 3.11(c)). Thus:

$$(p_1 - p_2)A = ma = m\frac{dv}{dt} \qquad [21]$$

where a is the acceleration and so the rate of change of velocity v. The mass of fluid being accelerated is $m = AL\rho$ and the rate of flow $q = Av$ and so:

$$(p_1 - p_2)A = L\rho\frac{dq}{dt}$$

$$p_1 - p_2 = I\frac{dq}{dt} \qquad [22]$$

where the inertance I is given by $I = L\rho/A$.

Example

Develop a model for the hydraulic system (Figure 3.12) where there is a liquid entering a container at one rate q_1 and leaving through a valve at another rate q_2.

We can neglect the inertance since flow rates can be assumed to change only very slowly. For the capacitance term we have:

$$q_1 - q_2 = C\frac{dp}{dt} = \frac{A}{\rho g}\frac{dp}{dt}$$

For the resistance term for the valve we have $p_1 - p_2 = Rq_1$. Thus, substituting for q_2, and recognising that the pressure difference is $h\rho g$, gives:

$$q_1 = A\frac{dh}{dt} + \frac{h\rho g}{R}$$

Problems 3.1

1 Propose a mathematical model for the oscillations of a suspension bridge when subject to wind gusts.
2 Propose a mathematical model for a machine mounted on firm ground when the machine is subject to forces when considered in terms of lumped-parameters.

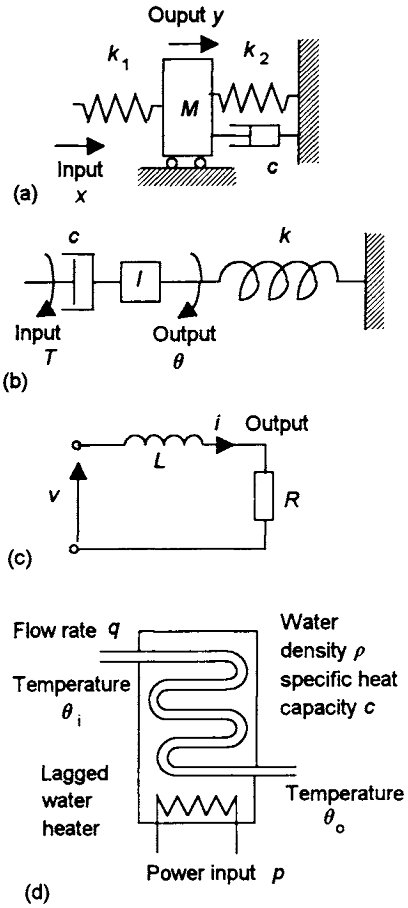


Figure 3.13 *Problem 3*

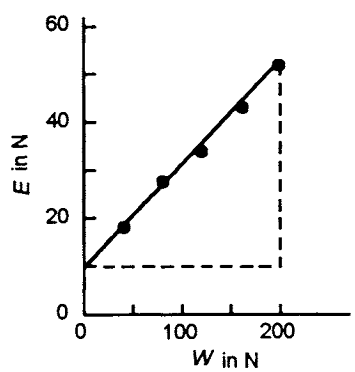


Figure 3.14 *Example*

3 Derive an equation for a mathematical model relating the input and output for each of the lumped systems shown in Figure 3.13.

3.2 Relating models and data

In testing mathematical models against real data, we often have the situation of having to check whether data fits an equation. If the relationship is linear, i.e. of the form $y = mx + c$, then it is comparatively easy to see whether the data fits the straight line and to ascertain the gradient m and intercept c. However, if the relationship is non-linear this is not so easy. A technique which can be used is to turn the non-linear equation into a linear one by changing the variables. Thus, if we have a relationship of the form $y = ax^2 + b$, instead of plotting y against x to give a non-linear graph we can plot y against x^2 to give a linear graph with gradient a and intercept b. If we have a relationship of the form $y = a/x$ we can plot a graph of y against $1/x$ to give a linear graph with a gradient of a.

Example

The following data was obtained from measurements of the load lifted by a machine and the effort expended. Determine if the relationship between the effort E and the load W is linear and if so the relationship.

E in N	18	27	32	43	51
W in N	40	80	120	160	200

Within the limits of experimental error the results appear to indicate a straight-line relationship (Figure 3.14). The gradient is 41/200 or about 0.21. The intercept with the E axis is at 10. Thus the relationship is $E = 0.21W + 10$.

Example

It is believed that the relationship between y and x for the following data is of the form $y = ax^2 + b$. Determine the values of a and b.

y	2.5	4.0	6.5	10.0	14.5
x	1	2	3	4	5

Figure 3.15 shows the graph of y against x^2. The graph has a gradient of AB/BC = 12.5/25 = 0.5 and an intercept with the y-axis of 2. Thus the relationship is $y = 0.5x^2 + 2$.

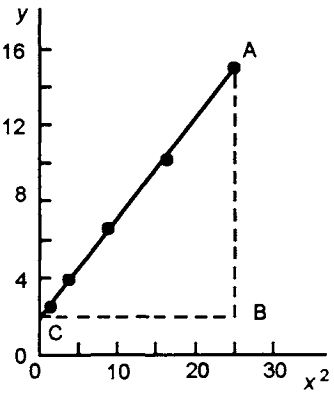


Figure 3.15 *Example*

Problems 3.2

1 Determine, assuming linear, the relationships between the following variables:

(a) The load L lifted by a machine for the effort E applied.

E in N	9.5	11.8	14.1	16.3	18.5
L in N	10	15	20	25	30

(b) The resistance R of a wire for different lengths L of that wire.

R in W	2.1	4.3	6.3	8.3	10.5
L in m	0.5	1.0	1.5	2.0	2.5

2 Determine what form the variables in the following equations should take when plotted in order to give straight-line graphs and what the values of the gradient and intercept will have.

(a) The period of oscillation T of a pendulum is related to the length L of the pendulum by the equation:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where g is a constant.
(b) The distance s travelled by a uniformly accelerating object after a time t is given by the equation:

$$s = ut + \tfrac{1}{2}at^2$$

where u and a are constants.
(c) The e.m.f. e generated by a thermocouple at a temperature θ is given by the equation;

$$e = a\theta + b\theta^2$$

where a and b are constants.
(d) The resistance R of a resistor at a temperature h is given by the equation:

$$R = R_0 + R_0 a\theta$$

where R_0 and a are constants.
(f) The pressure p of a gas and its volume V are related by the equation:

$$pV = k$$

where k is a constant.

(g) The deflection y of the free end of a cantilever due to it own weight of w per unit length is related to its length L by the equation:

$$y = \frac{wL^4}{8EI}$$

where w, E and I are constants.

3 The resistance R of a lamp is measured at a number of voltages V and the following data obtained. Show that the law relating the resistance to the voltage is of the form $R = (a/V) + b$ and determine the values of a and b.

R in Ω	70	62	59	56	55
V in V	60	100	140	200	240

4 The resistance R of wires of a particular material are measured for a range of wire diameters d and the following results obtained. Show that the relationship is of the form $R = (a/d^2) + b$ and determine the values of a and b.

R in Ω	0.25	0.16	0.10	0.06	0.04
d in mm	0.80	1.00	1.25	1.60	2.00

5 The volume V of a gas is measured at a number of pressures p and the following results obtained. Show that the relationship is of the form $V = ap^b$ and determine the values of a and b.

V in m^3	13.3	11.4	10.0	8.9	8.0
p in 10^5 Pa	1.2	1.4	1.6	1.8	2.0

6 When a gas is compressed adiabatically the pressure p and temperature T are measured and the following results obtained. Show that the relationship is of the form $T = ap^b$ and determine the values of a and b.

p in 10^5 Pa	1.2	1.5	1.8	2.1	2.4
T in K	526	560	589	615	639

7 The cost C per hour of operating a machine depends on the number of items n produced per hour. The following data has been obtained and is anticipated to follow a relationship of the form $C = an^3 + b$. Show that this is the case and determine the values of a and b.

C in £	31	38	67	94	155
n	10	20	30	40	50

8 The following are suggested braking distances s for cars travelling at different speeds v. The relationship between s and v is thought to be of the form $s = av^2 + bv$. Show that this is so and determine the values of a and b.

s in m	5	15	30	50	75
v in m/s	5	10	15	20	25

Hint: consider s/v as one of the variables.

9 The luminosity I of a lamp depends on the voltage V applied to it. The relationship between I and V is thought to be of the form $I = aV^b$. Use the following results to show that this is the case and determine the values of a and b.

I in candela	3.6	6.4	10.0	14.4	19.6
V in volts	60	80	100	120	140

10 From a lab test, it is believed that the law relating the voltage v across an inductor and the time t is given by the relationship $v = A\ e^{t/B}$, where A and B are constant and e is the exponential function. From the lab test the results observed were:

v (volts)	908.4	394.8	171.6	32.4	14.1	6.12
t (ms)	10	20	30	50	60	70

Show that the law relating the voltage to time is, in fact, true. Then determine the values of the constants A and B.